ANSWER on Question #72277 Math. Geometry

Compute the curvature of the following curves

$$1)\ \gamma(t) = \left(\frac{4}{5}\cos t\,, 1 - \sin t\,, -\frac{3}{5}\cos t\right)$$

$$2)\ \gamma(t) = (t, \cosh t)$$

$$3)\ \gamma(t) = \frac{4}{5}(\cos^3 t\,, \sin^3 t)$$

For the astroid in (3), show that the curvature tends to ∞ as we approach one of the points $(\pm 1,0)$, $(0, \pm 1)$

SOLUTION

By the definition, we want to briefly discuss the **curvature** of a smooth curve (recall that for a smooth curve we require $\overrightarrow{r'(t)}$ is continuous and $\overrightarrow{r'(t)} \neq 0$). The curvature measures how fast a curve is changing direction at a given point.

There are several formulas for determining the curvature for a curve. The formal definition of curvature is,

$$k = \left|\frac{d\vec{T}}{ds}\right|$$

Where $\vec{T}$ the unit tangent and s is the arc length.

In general the formal definition of the curvature is not easy to use so there are two alternate formulas that we can use. Here they are.

$$k = \frac{\left|\overrightarrow{T'(t)}\right|}{\left|\overrightarrow{r'(t)}\right|}$$

$$k = \frac{\left|\overrightarrow{r'(t)} \times \overrightarrow{r''(t)}\right|}{\left|\overrightarrow{r'(t)}\right|^3}$$

In our case,

$$1)\ \gamma(t) = \left(\frac{4}{5}\cos t\,, 1 - \sin t\,, -\frac{3}{5}\cos t\right)$$

$$\gamma'(t) = \left(-\frac{4}{5}\sin t\,, -\cos t\,, \frac{3}{5}\sin t\right)$$

$$|\gamma'(t)| = \sqrt{\left(-\frac{4}{5}\sin t\right)^2 + (-\cos t)^2 + \left(\frac{3}{5}\sin t\right)^2} = \sqrt{\frac{15}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t} =$$

$$= \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1 \rightarrow \boxed{|\gamma'(t)| = 1}$$

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{\left(-\frac{4}{5}\sin t\,, -\cos t\,, \frac{3}{5}\sin t\right)}{1} = \left(-\frac{4}{5}\sin t\,, -\cos t\,, \frac{3}{5}\sin t\right)$$

$$T'(t) = \left(-\frac{4}{5}\cos t\,, \sin t\,, \frac{3}{5}\cos t\right)$$

$$|T'(t)| = \sqrt{\left(-\frac{4}{5}\cos t\right)^2 + (\sin t)^2 + \left(\frac{3}{5}\cos t\right)^2} = \sqrt{\frac{15}{25}\cos^2 t + \sin^2 t + \frac{9}{25}\cos^2 t} =$$

$$= \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1 \rightarrow \boxed{|T'(t)| = 1}$$

Then,

$$k = \frac{|T'(t)|}{|\gamma'(t)|} = \frac{1}{1} = 1$$

Conclusion,

$$\boxed{\gamma(t) = \left(\frac{4}{5}\cos t\,, 1 - \sin t\,, -\frac{3}{5}\cos t\right) \rightarrow k = 1}$$

$$2)\ \gamma(t) = (t, \cosh t)$$

$$\gamma'(t) = (1, \sinh t)$$

$$|\gamma'(t)| = \sqrt{(1)^2 + (\sinh t)^2} = \begin{bmatrix} \cosh^2 t - \sinh^2 t = 1 \rightarrow \\ \cosh^2 t = 1 + \sinh^2 t \end{bmatrix} = \sqrt{\cosh^2 t} = \cosh t$$

$$\boxed{|\gamma'(t)| = \cosh t}$$

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{(1, \sinh t)}{\cosh t} = \left(\frac{1}{\cosh t}, \tanh t\right)$$

$$T'(t) = \left(-\frac{\sinh t}{\cosh^2 t}, \frac{1}{\cosh^2 t}\right)$$

$$|T'(t)| = \sqrt{\left(-\frac{\sinh t}{\cosh^2 t}\right)^2 + \left(\frac{1}{\cosh^2 t}\right)^2} = \sqrt{\frac{1 + \sinh^2 t}{\cosh^4 t}} = \begin{bmatrix} \cosh^2 t - \sinh^2 t = 1 \rightarrow \\ \cosh^2 t = 1 + \sinh^2 t \end{bmatrix} =$$

$$= \sqrt{\frac{\cosh^2 t}{\cosh^4 t}} = \sqrt{\frac{1}{\cosh^2 t}} = \frac{1}{\cosh t} \rightarrow \boxed{|T'(t)| = \frac{1}{\cosh t}}$$

Then,

$$k = \frac{|T'(t)|}{|\gamma'(t)|} = \frac{\frac{1}{\cosh t}}{\cosh t} = \frac{1}{\cosh^2 t}$$

Conclusion,

$$\boxed{\gamma(t) = \gamma(t) = (t, \cosh t) \rightarrow k = \frac{1}{\cosh^2 t}}$$

$$3)\ \gamma(t)=\frac{4}{5}(\cos^3 t\,,\sin^3 t)$$

$$\gamma'(t)=\frac{4}{5}(3\cos^2 t\,(-\sin t),3\sin^2 t\cos t)$$

$$\gamma'(t)=\frac{2\cdot 3}{5}\big((-\cos t)(2\cos t\sin t),\sin t\,(2\sin t\cos t)\big)$$

$$\gamma'(t)=\frac{6}{5}\big((-\cos t)\sin 2t\,,\sin t\sin 2t\big)$$

$$|\gamma'(t)|=\frac{4}{5}\sqrt{\big(3\cos^2 t\,(-\sin t)\big)^2+(3\sin^2 t\cos t)^2}=$$

$$=\frac{4}{5}\cdot 3\sqrt{\cos^4 t\sin^2 t+\sin^4 t\cos^2 t}=\frac{12}{5}\sqrt{\cos^2 t\sin^2 t\,(\cos^2 t+\sin^2 t)}=$$

$$=\frac{12}{5}\sqrt{\cos^2 t\sin^2 t}=\frac{12}{5}\cos t\sin t=\frac{6}{5}\cdot(2\cos t\sin t)=\frac{6}{5}\cdot\sin 2t$$

$$\boxed{|\gamma'(t)|=\frac{6}{5}\cdot\sin 2t}$$

$$T(t)=\frac{\gamma'(t)}{|\gamma'(t)|}=\frac{\frac{6}{5}\big((-\cos t)\sin 2t\,,\sin t\sin 2t\big)}{\frac{6}{5}\cdot\sin 2t}=(-\cos t\,,\sin t)$$

$$T'(t)=(\sin t\,,\cos t)$$

$$|T'(t)|=\sqrt{(\sin t)^2+(\cos t)^2}=\sqrt{\cos^2 t+\sin^2 t}=1$$

$$\boxed{|T'(t)|=1}$$

Then,

$$k=\frac{|T'(t)|}{|\gamma'(t)|}=\frac{1}{\frac{6}{5}\cdot\sin 2t}=\frac{5}{6\sin 2t}$$

Conclusion,

$$\boxed{\gamma(t) = \gamma(t) = \frac{4}{5}(\cos^3 t\,, \sin^3 t) \to k = \frac{5}{6 \sin 2t}}$$

For the astroid in (3), show that the curvature tends to ∞ as we approach one of the points $(\pm 1{,}0)$, $(0, \pm 1)$

If we approach the point $(1{,}0)$ - this means that $t = 0$

Then,

$$\lim_{t \to 0} k = \lim_{t \to 0} \frac{5}{6 \sin 2t} = \frac{5}{6 \cdot \sin(2 \cdot 0)} = \frac{5}{6 \cdot \sin 0} = \frac{5}{6 \cdot 0} = \infty$$

If we approach the point $(0{,}1)$ - this means that $t = {}^{\pi}/_{2}$

$$\lim_{t \to \frac{\pi}{2}} k = \lim_{t \to \frac{\pi}{2}} \frac{5}{6 \sin 2t} = \frac{5}{6 \cdot \sin\left(2 \cdot \frac{\pi}{2}\right)} = \frac{5}{6 \cdot \sin \pi} = \frac{5}{6 \cdot 0} = \infty$$