# Answer on Question #72097 – Math – Calculus

## Question

Obtain the Fourier series for the following periodic function which has period of $2\pi$:
$f(x) = x^2$ for $-\pi \le x \le \pi$

## Solution

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ , -\pi \le x \le \pi$$

$f(-x) = (-x)^2 = x^2 = f(x) => f(x)$ is even function
Then we have Fourier Cosine Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \ , -\pi \le x \le \pi$$

$b_n = 0$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx$$

$$a_0 = \frac{1}{\pi}\int_{0}^{\pi} x^2\, dx = \frac{1}{3\pi} x^3 \Big|_0^{\pi} = \frac{1}{3\pi}(\pi^3 - 0) = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\, dx = \frac{2}{\pi}\left(\frac{1}{n}x^2 \sin nx + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right) \Big|_0^{\pi} =$$

$$= \frac{2}{\pi}\left(\frac{1}{n}\pi^2 \sin n\pi + \frac{2}{n^2}\pi \cos n\pi - \frac{2}{n^3}\sin n\pi\right) - \frac{2}{\pi}(0 + 0 - 0) =$$

$$= \frac{2}{\pi}\left(0 + \frac{2}{n^2}\pi(-1)^n - 0\right) = (-1)^n \frac{4}{n^2}$$

$$I = \int x^2 \cos nx\, dx = \frac{1}{n}x^2 \sin nx - \frac{2}{n}\int x \sin nx\, dx =$$

$$= \frac{1}{n}x^2 \sin nx + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx$$

$$\int u\, dv = uv - \int v\, du$$

$u = x^2, du = 2xdx$

$dv = \cos nx\, dx, v = \frac{1}{n}\sin nx$

$\int x \sin nx\, dx = -\frac{1}{n} x \cos nx + \frac{1}{n}\int \cos nx\, dx = -\frac{1}{n} x \cos nx + \frac{1}{n^2}\sin nx$

$$\int u\, dv = uv - \int v\, du$$

$u = x, du = dx$

$dv = \sin nx\, dx, v = -\frac{1}{n}\cos nx$

$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}\cos nx$

**Answer:** $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}\cos nx$