

QUESTION

Compute the curvature of the following curves

$$1) \gamma(t) = \left(\frac{4}{5} \cos t, 1 - \sin t, -\frac{3}{5} \cos t \right)$$

$$2) \gamma(t) = (t, \cosh t)$$

$$3) \gamma(t) = \frac{4}{5} (\cos^3 t, \sin^3 t)$$

For the astroid in (3), show that the curvature tends to ∞ as we approach one of the points $(\pm 1, 0), (0, \pm 1)$

SOLUTION

By the definition, we want to briefly discuss the **curvature** of a smooth curve (recall that for a smooth curve we require $\overrightarrow{r'(t)}$ is continuous and $\overrightarrow{r'(t)} \neq 0$). The curvature measures how fast a curve is changing direction at a given point.

There are several formulas for determining the curvature for a curve. The formal definition of curvature is,

$$k = \left| \frac{d\vec{T}}{ds} \right|$$

Where $\vec{T}$ the unit tangent and s is the arc length.

In general the formal definition of the curvature is not easy to use so there are two alternate formulas that we can use. Here they are.

$$k = \frac{|\overrightarrow{T'(t)}|}{|\overrightarrow{r'(t)}|}$$

$$k = \frac{|\overrightarrow{r'(t)} \times \overrightarrow{r''(t)}|}{|\overrightarrow{r'(t)}|^3}$$

In our case,

1)

$$\gamma(t) = \left(\frac{4}{5} \cos t, 1 - \sin t, -\frac{3}{5} \cos t \right)$$

$$\gamma'(t) = \left(-\frac{4}{5} \sin t, -\cos t, \frac{3}{5} \sin t \right)$$

$$\begin{aligned} |\gamma'(t)| &= \sqrt{\left(-\frac{4}{5} \sin t\right)^2 + (-\cos t)^2 + \left(\frac{3}{5} \sin t\right)^2} = \sqrt{\frac{15}{25} \sin^2 t + \cos^2 t + \frac{9}{25} \sin^2 t} = \\ &= \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1 \rightarrow \boxed{|\gamma'(t)| = 1} \end{aligned}$$

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{\left(-\frac{4}{5} \sin t, -\cos t, \frac{3}{5} \sin t\right)}{1} = \left(-\frac{4}{5} \sin t, -\cos t, \frac{3}{5} \sin t\right)$$

$$T'(t) = \left(-\frac{4}{5} \cos t, \sin t, \frac{3}{5} \cos t\right)$$

$$\begin{aligned} |T'(t)| &= \sqrt{\left(-\frac{4}{5} \cos t\right)^2 + (\sin t)^2 + \left(\frac{3}{5} \cos t\right)^2} = \sqrt{\frac{16}{25} \cos^2 t + \sin^2 t + \frac{9}{25} \cos^2 t} = \\ &= \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1 \rightarrow \boxed{|T'(t)| = 1} \end{aligned}$$

Then,

$$k = \frac{|T'(t)|}{|\gamma'(t)|} = \frac{1}{1} = 1$$

Conclusion,

$$\boxed{\gamma(t) = \left(\frac{4}{5} \cos t, 1 - \sin t, -\frac{3}{5} \cos t\right) \rightarrow k = 1}$$

2)

$$\gamma(t) = (t, \cosh t)$$

$$\gamma'(t) = (1, \sinh t)$$

$$|\gamma'(t)| = \sqrt{(1)^2 + (\sinh t)^2} = \left[\begin{array}{l} \cosh^2 t - \sinh^2 t = 1 \rightarrow \\ \cosh^2 t = 1 + \sinh^2 t \end{array} \right] = \sqrt{\cosh^2 t} = \cosh t$$

$$\boxed{|\gamma'(t)| = \cosh t}$$

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{(1, \sinh t)}{\cosh t} = \left(\frac{1}{\cosh t}, \tanh t \right)$$

$$T'(t) = \left(-\frac{\sinh t}{\cosh^2 t}, \frac{1}{\cosh^2 t} \right)$$

$$\begin{aligned} |T'(t)| &= \sqrt{\left(-\frac{\sinh t}{\cosh^2 t} \right)^2 + \left(\frac{1}{\cosh^2 t} \right)^2} = \sqrt{\frac{1 + \sinh^2 t}{\cosh^4 t}} = \left[\begin{array}{l} \cosh^2 t - \sinh^2 t = 1 \rightarrow \\ \cosh^2 t = 1 + \sinh^2 t \end{array} \right] = \\ &= \sqrt{\frac{\cosh^2 t}{\cosh^4 t}} = \sqrt{\frac{1}{\cosh^2 t}} = \frac{1}{\cosh t} \rightarrow \boxed{|T'(t)| = \frac{1}{\cosh t}} \end{aligned}$$

Then,

$$k = \frac{|T'(t)|}{|\gamma'(t)|} = \frac{\frac{1}{\cosh t}}{\cosh t} = \frac{1}{\cosh^2 t}$$

Conclusion,

$$\boxed{\gamma(t) = \gamma(t) = (t, \cosh t) \rightarrow k = \frac{1}{\cosh^2 t}}$$

3)

$$\gamma(t) = \frac{4}{5}(\cos^3 t, \sin^3 t)$$

$$\gamma'(t) = \frac{4}{5}(3 \cos^2 t (-\sin t), 3 \sin^2 t \cos t)$$

$$\gamma'(t) = \frac{2 \cdot 3}{5}((- \cos t)(2 \cos t \sin t), \sin t (2 \sin t \cos t))$$

$$\gamma'(t) = \frac{6}{5}((- \cos t) \sin 2t, \sin t \sin 2t)$$

$$\begin{aligned} |\gamma'(t)| &= \frac{4}{5} \sqrt{(3 \cos^2 t (-\sin t))^2 + (3 \sin^2 t \cos t)^2} = \\ &= \frac{4}{5} \cdot 3 \sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} = \frac{12}{5} \sqrt{\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} = \\ &= \frac{12}{5} \sqrt{\cos^2 t \sin^2 t} = \frac{12}{5} \cos t \sin t = \frac{6}{5} \cdot (2 \cos t \sin t) = \frac{6}{5} \cdot \sin 2t \end{aligned}$$

$$\boxed{|\gamma'(t)| = \frac{6}{5} \cdot \sin 2t}$$

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{\frac{6}{5}((- \cos t) \sin 2t, \sin t \sin 2t)}{\frac{6}{5} \cdot \sin 2t} = (- \cos t, \sin t)$$

$$T'(t) = (\sin t, \cos t)$$

$$|T'(t)| = \sqrt{(\sin t)^2 + (\cos t)^2} = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\boxed{|T'(t)| = 1}$$

Then,

$$k = \frac{|T'(t)|}{|\gamma'(t)|} = \frac{1}{\frac{6}{5} \cdot \sin 2t} = \frac{5}{6 \sin 2t}$$

Conclusion,

$$\boxed{\gamma(t) = \gamma(t) = \frac{4}{5}(\cos^3 t, \sin^3 t) \rightarrow k = \frac{5}{6 \sin 2t}}$$

For the astroid in (3), show that the curvature tends to ∞ as we approach one of the points $(\pm 1, 0)$, $(0, \pm 1)$

If we approach the point $(1, 0)$ - this means that $t = 0$

Then,

$$\lim_{t \rightarrow 0} k = \lim_{t \rightarrow 0} \frac{5}{6 \sin 2t} = \frac{5}{6 \cdot \sin(2 \cdot 0)} = \frac{5}{6 \cdot \sin 0} = \frac{5}{6 \cdot 0} = \infty$$

If we approach the point $(0, 1)$ - this means that $t = \pi/2$

$$\lim_{t \rightarrow \frac{\pi}{2}} k = \lim_{t \rightarrow \frac{\pi}{2}} \frac{5}{6 \sin 2t} = \frac{5}{6 \cdot \sin\left(2 \cdot \frac{\pi}{2}\right)} = \frac{5}{6 \cdot \sin \pi} = \frac{5}{6 \cdot 0} = \infty$$

