Answer on Question #71820 – Math – Discrete Mathematics

Question

Prove that for any integer n such that n≡2 mod 3 , n^k≡1 or 2 (mod 3) , where k∈N

Solution

From $n \equiv 2 \, mod \, 3$ we can write n as $n = 3t + 2$, where t is integer number.
Consider

$$n^k = (3t + 2)^k.$$

Using formula

$$(a + b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i,$$

$$(3t + 2)^k = \binom{k}{0}(3t)^k + \binom{k}{1}(3t)^{k-1}2^1 + \cdots + \binom{k}{k-1}(3t)^1 2^{k-1} + \binom{k}{k} 2^k.$$

Obviously,

$$\left(\binom{k}{0}(3t)^k + \binom{k}{1}(3t)^{k-1}2^1 + \cdots + \binom{k}{k-1}(3t)^1 2^{k-1}\right) \equiv 0 \, mod \, 3$$

because each term of the sum has a multiplier of 3.
Therefore,

$$n^k \equiv (3t + 2)^k \equiv 2^k \, mod \, 3.$$

Using the principle of mathematical induction verify statement

$$2^k \equiv 1 \, mod \, 3 \text{ or } 2^k \equiv 2 \, mod \, 3 \text{ for } k \in N.$$

For $k = 1$ and $k = 2$ statement is true:

$$2^1 \equiv 2 \, mod \, 3, \, 2^2 \equiv 4 \equiv 1 \, mod \, 3.$$

Let the statement be true for $k = l$:

$$2^l \equiv 1 \, mod \, 3 \text{ or } 2^l \equiv 2 \, mod \, 3.$$

Consider $k = l + 1$.
If $2^l \equiv 1 \, mod \, 3$ then $2^{l+1} \equiv 2 \cdot 2^l \equiv 2 \cdot 1 \equiv 2 \, mod \, 3$.
If $2^l \equiv 2 \, mod \, 3$ then $2^{l+1} \equiv 2 \cdot 2^l \equiv 2 \cdot 2 \equiv 4 \equiv 1 \, mod \, 3$.
Hence the statement $2^{l+1} \equiv 2 \, mod \, 3$ or $2^{l+1} \equiv 1 \, mod \, 3$ is true.
According to the principle of mathematical induction the statement is true for $k \in N$.
As a result, one gets $n^k \equiv 2^k \equiv 1 \, mod \, 3$ or $n^k \equiv 2^k \equiv 2 \, mod \, 3$.