Answer on Question #71393, Math / Functional Analysis

Is the real line a metric space?

Solution. Yes, the real line is a metric space.

Recall that a set X is a metric space if there is a function $d: X^2 \to \mathrm{R}$ such that for each $x, y, z \in X$:

(M1) $d(x, y) \geq 0$;

(M2) $d(x, y) = d(y, x)$;

(M3) $d(x, y) \leq d(x, z) + d(z, y)$;

(M4) $d(x, y) = 0 \Leftrightarrow x = y$.

Let $d: \mathrm{R}^2 \to \mathrm{R}$ be defined as:

$$d(x, y) = |x - y| \text{ for each } x, y \in \mathrm{R}.$$

(M1) $d(x, y) \geq 0$ because for each $x, y \in \mathrm{R}$.

(M2) $d(x, y) = |x - y| = |-(y - x)| = |y - x| = d(y, x)$;

(M3) $d(x, y) = |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$;

(M4) $d(x, y) = 0 \Leftrightarrow |x - y| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$.

Hence, the real line is a metric space.