Answer on Question #71276 – Math – Calculus

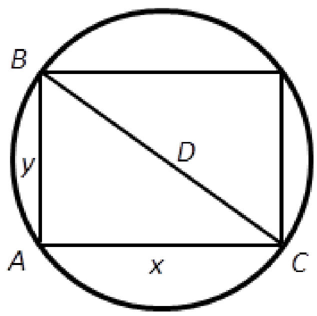
Question

Find the largest rectangle that can be inscribed in a given circle with radius of 12 ft.

Aguitar with trapezoidal cross section is to be made from a long sheet of stainless steel that is 30 cm wide by turning up one fourth of the width in each side. What width across the top will give the max cross-section area?

Solution

1. Find the largest rectangle that can be inscribed in a given circle with radius of 12 ft. To make a mathematical model we draw a diagram and label its parts. The quantity we need to maximize is the area of the rectangle which is given by



$$A = xy$$

In the inscribed rectangle the diagonal is the diameter of the circle that is BC is the diameter. Since the radius is 12 ft then the diameter of the circle is 24 ft. Using the Pythagorean Theorem with ΔABC we have

$$x^2 + y^2 = D^2 = 24^2 = 576$$

then we get

$$y = \sqrt{576 - x^2}$$

The area of the rectangle becomes

$$A = x\sqrt{576 - x^2}$$

In order to find the maximum area, we calculate the derivative of A with respect to x

$$A' = \left(x\sqrt{576 - x^2}\right)' = (x)'\sqrt{576 - x^2} + x\left(\sqrt{576 - x^2}\right)'$$

$$= \sqrt{576 - x^2} + x \cdot \frac{(-2x)}{2\sqrt{576 - x^2}} = \frac{576 - x^2 - x^2}{\sqrt{576 - x^2}} = \frac{576 - 2x^2}{\sqrt{576 - x^2}}$$

and equate it to zero

$$A' = \frac{576 - 2x^2}{\sqrt{576 - x^2}} = 0$$

Solving this equation we get

$$x^2 = 288$$

or

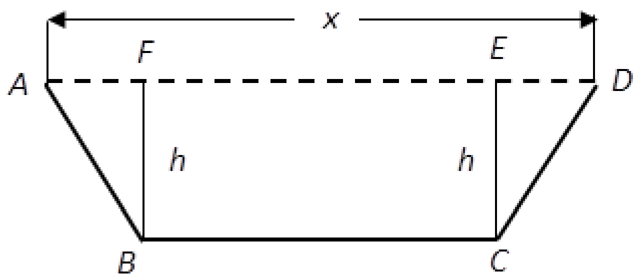
$$x = \sqrt{288} = \sqrt{2 \cdot 144} = 12\sqrt{2}$$

$x = 12\sqrt{2}$ is a stationary point in which the function $A = A(x)$ has a maximum. Then we find y:

$$y = \sqrt{576 - x^2} = \sqrt{576 - 288} = \sqrt{288} = 12\sqrt{2} = x$$

Answer 1. Thus largest rectangle that can be inscribed in a given circle with radius of 12 ft is the square with side $12\sqrt{2}\ ft$

2. To make a mathematical model of a gutter with trapezoidal cross section we draw a diagram and label its parts



By the condition of problem $AB = CD = \frac{30}{4} = 7.5\ \ cm$ then $BC = 30 - 2 \cdot 7.5 = 15\ \ cm.$

Area of trapezoid is

$$A = \frac{BC + AD}{2} \cdot h = \frac{1}{2}(15 + x)h$$

Express h in terms of x. It follows from the figure that

$$FE = BC = 15\ \ cm$$

Then

$$AF = ED = \frac{x - 15}{2}$$

Using the Pythagorean Theorem with ΔAFB we have

$$h = \sqrt{AB^2 - AF^2} = \sqrt{7.5^2 - \left(\frac{x - 15}{2}\right)^2} = \sqrt{56.25 - \frac{1}{4}(x^2 - 30x + 225)}$$

$$= \frac{1}{2}\sqrt{225 - x^2 + 30x - 225} = \frac{1}{2}\sqrt{30x - x^2}$$

The area of the trapezoid becomes

$$A = \frac{1}{4}(15 + x)\sqrt{30x - x^2}$$

In order to find the maximum area, we calculate the derivative of A with respect to x

$$A' = \left(\frac{1}{4}(15 + x)\sqrt{30x - x^2}\right)' = \frac{1}{4}(15 + x)'\sqrt{30x - x^2} + \frac{1}{4}(15 + x)\left(\sqrt{30x - x^2}\right)'$$

$$= \frac{1}{4}\sqrt{30x - x^2} + \frac{1}{4}(15 + x)\frac{30 - 2x}{2\sqrt{30x - x^2}}$$

$$= \frac{1}{4}\sqrt{30x - x^2} + \frac{1}{4}(15 + x)\frac{15 - x}{\sqrt{30x - x^2}}$$

$$= \frac{30x - x^2 + (15 + x)(15 - x)}{4\sqrt{30x - x^2}} = \frac{30x - x^2 + 225 - x^2}{4\sqrt{30x - x^2}}$$

$$= \frac{-2x^2 + 30x + 225}{4\sqrt{30x - x^2}}$$

Equate it to zero and find the stationary points

$$A' = \frac{-2x^2 + 30x + 225}{4\sqrt{30x - x^2}} = 0$$

or

$$2x^2 - 30x - 225 = 0$$

Solve this equation

$$x = \frac{30 \pm \sqrt{900 - 4 \cdot 2 \cdot (-225)}}{2 \cdot 2} = \frac{30 \pm \sqrt{900 + 1800}}{4} = \frac{30 \pm \sqrt{2700}}{4} = \frac{30 \pm 30\sqrt{3}}{4}$$

We get

$x_1 = 7.5 \cdot \left(1 + \sqrt{3}\right) \approx 20.5\ \ cm$

$x_2 = 7.5 \cdot \left(1 - \sqrt{3}\right) < 0$

Since width cannot be negative, only x_1 should be considered.

Answer 2. Width across the top of trapezoidal cross section which will give the max cross-section area is

$$x = 7.5 \cdot \left(1 + \sqrt{3}\right) \approx 20.5\ \ cm$$