Answer on Question #71146 – Math – Real Analysis

Question

Is $d(x,y)=\sqrt{|x-y|}$ a metric?

Solution

First let us recall the definition of the metric

(see https://en.wikipedia.org/wiki/Metric_(mathematics)#Definition).

Let us assume that $x,y\in\mathbb{R}$. Now let us check all four conditions from definition.

1. $d(x,y)\geq 0$. Since $|x-y|\geq 0$ due to the property of absolute value, and the square root is also nonnegative function we conclude that the condition 1 is satisfied.

2. $d(x,y)=0\Leftrightarrow\sqrt{|x-y|}=0\Leftrightarrow|x-y|=0\Leftrightarrow x=y$, so the second condition is satisfied.

3. $d(x,y)=\sqrt{|x-y|}=\sqrt{|y-x|}=d(y,x)$, and the condition of symmetry is also satisfied.

4. $d(x,z)\leq d(x,y)+d(y,z)$. Let us start from the well-known triangle inequality

(see https://en.wikipedia.org/wiki/Triangle_inequality#Example_norms)

$|a+b|\leq|a|+|b|$. Let us put $a\coloneqq x-y, b\coloneqq y-z\Rightarrow a+b=x-z$. Then we get:

$|x-z|\leq|x-y|+|y-z|$. The last inequality is easily transformed to

$|x-z|\leq|x-y|+|y-z|+2\sqrt{|x-y|\cdot|y-z|}\Leftrightarrow|x-z|\leq\left(\sqrt{|x-y|}+\sqrt{|y-z|}\right)^2\Leftrightarrow$

$\Leftrightarrow\sqrt{|x-z|}\leq\sqrt{|x-y|}+\sqrt{|y-z|}$, so we obtained the required inequality

$d(x,z)\leq d(x,y)+d(y,z)$, and the last condition is also satisfied.

We see that all four conditions from definition are satisfied so $d(x,y)=\sqrt{|x-y|}$ is a metric.

Answer: Yes.