Answer on Question #71135 – Math – Differential Equations

Question

Solve the partial differential equation p cos(x+y) + q sin(x+y)=z, where the partial derivatives of z with respect to x,y are denoted by p and q respectively.?

Solution

$$\frac{\partial z}{\partial x}\cos(x+y)+\frac{\partial z}{\partial y}\sin(x+y)=z$$

$$\frac{dx}{\cos(x+y)}=\frac{dy}{\sin(x+y)}=\frac{dz}{z}$$

$$\begin{cases}\dfrac{dy}{\sin(x+y)}=\dfrac{dz}{z}\\ \dfrac{dx}{\cos(x+y)}=\dfrac{dz}{z}\end{cases}$$

Solve the first equation of the system:

$$\int\frac{dy}{\sin(x+y)}=\int\frac{dz}{z}$$

$\int\frac{dy}{\sin(x+y)}=\int\frac{\sin(x+y)}{1-(\cos(x+y))^2}dx=\{let\ u=\cos(x+y)\ and\ du=-\sin(x+y)\,dx\}=$
$\int\frac{-1}{1+u^2}du=\frac{-1}{2}\int\frac{1}{1+u}du-\frac{1}{2}\int\frac{1}{1-u}du=\frac{\ln(u-1)}{2}-\frac{\ln(u+1)}{2}+Const=\frac{\ln(\cos(x+y)-1)}{2}-$
$-\frac{\ln(\cos(x+y)+1)}{2}+Const,$

where $Const$ is an integration constant.

$$\frac{\ln(\cos(x+y)-1)}{2}-\frac{\ln(\cos(x+y)+1)}{2}+Const=ln|z|$$

$$z=\pm\frac{\sqrt{\cos(x+y)-1}}{\sqrt{\cos(x+y)+1}}e^{Const}$$

$$z=C1\sqrt{\frac{\cos(x+y)-1}{\cos(x+y)+1}}$$

Solve the second equation of the system:

$$\int \frac{dx}{\cos(x+y)} = \int \frac{dz}{z}$$

$$\int \frac{dy}{\cos(x+y)} = \int \frac{\cos(x+y)}{1-(\sin(x+y))^2}\,dx = \{let\ u = \sin(x+y)\ and\ du = \cos(x+y)\,dx\}$$
$$= \int \frac{1}{1-u^2}\,du =$$

$$= \frac{1}{2}\int \frac{1}{1+u}\,du + \frac{1}{2}\int \frac{1}{1-u}\,du = \frac{\ln(u+1)}{2} - \frac{\ln(u-1)}{2} + Const = \frac{\ln(\sin(x+y)+1)}{2} - \frac{\ln(\sin(x+y)-1)}{2} + Const,$$

where $Const$ is an integration constant.

$$\frac{\ln(\sin(x+y)+1)}{2} - \frac{\ln(\sin(x+y)-1)}{2} + Const = ln|z|$$

$$z = \pm\frac{\sqrt{\sin(x+y)+1}}{\sqrt{\sin(x+y)-1}}e^{Const}$$

$$z = C2\sqrt{\frac{\sin(x+y)+1}{\sin(x+y)-1}}$$

Then

$$\begin{cases} C1 = z\sqrt{\dfrac{\cos(x+y)+1}{\cos(x+y)-1}} \\ C2 = z\sqrt{\dfrac{\sin(x+y)-1}{\sin(x+y)+1}} \end{cases}$$

The solution of the partial differential equation is given by

$$F\left(z\sqrt{\frac{\cos(x+y)+1}{\cos(x+y)-1}}; z\sqrt{\frac{\sin(x+y)-1}{\sin(x+y)+1}}\right) = 0,$$

where *F* is an arbitrary differentiable function.

Answer:

$$F\left(z\sqrt{\frac{\cos(x+y)+1}{\cos(x+y)-1}}; z\sqrt{\frac{\sin(x+y)-1}{\sin(x+y)+1}}\right) = 0,$$

where *F* is an arbitrary differentiable function.