Answer on Question # 70958 – Math – Calculus

Question

Find the volume of the solid of revolution obtained by rotating the curve

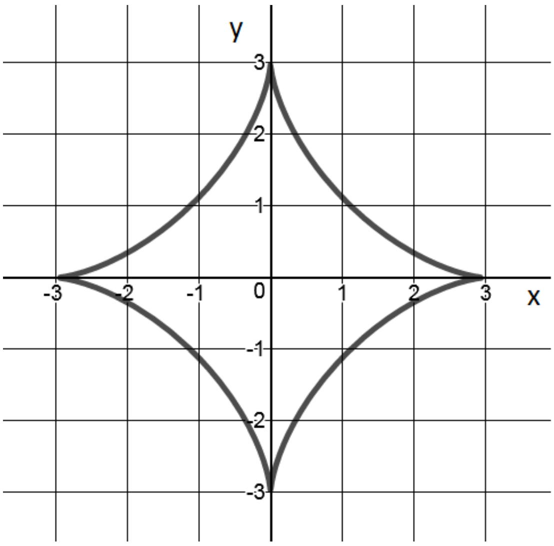
$x = 3\cos^3\theta$, $y = 3\sin^3\theta$ about the x-axis.

Solution

A curve defined parametrically by

$$x(\theta) = 3\cos^3\theta, \quad y(\theta) = 3\sin^3\theta \qquad (1)$$

is called an asteroid. It is shown in the figure below.



The parametric curve (1) is revolved about the x-axis. The volume of a solid of revolution is given by

$$V = \int_a^b \pi y^2(x)dx$$

For this case

$$y = y(\theta) = 3\sin^3\theta,$$

$$x = x(\theta) = 3\cos^3\theta,$$

x varies within $[-3, 3]$ so $a = -3$ and $b = 3$. We get

$$V = \int_{x=-3}^{x=3} \pi y^2(\theta)dx(\theta) = 2\int_{x=0}^{x=3} \pi y^2(\theta)dx(\theta) \qquad (2)$$

Find dx:

$$dx = d(3\cos^3\theta) = 3(\cos^3\theta)'d\theta = 3 \cdot 3\cos^2\theta \cdot (-\sin\theta)d\theta = -9\sin\theta\cos^2\theta d\theta$$

When $x = 0$, $\cos^3\theta = 0$ so $\cos\theta = 0$ and $\theta = \frac{\pi}{2}$;

when $x = 3$, $\cos^3\theta = 1$ so $\cos\theta = 1$ and $\theta = 0$.

Substitute the known values into (2):

$V = 2\int_{x=0}^{x=3} \pi(3\sin^3\theta)^2(-9\sin\theta\cos^2\theta d\theta) = -2\pi \cdot 81 \int_{x=0}^{x=3}(\sin^2\theta)^3\cos^2\theta\sin\theta d\theta.$

$$V = -162\pi \int_{\theta=\pi/2}^{\theta=0} (1-\cos^2\theta)^3\cos^2\theta\sin\theta d\theta$$

Substitute $\cos\theta = u$. We get $\sin\theta d\theta = d(-\cos\theta) = -du$;

if $\theta = \pi/2$, then $\cos(\pi/2) = 0$ so $u = 0$,

if $\theta = 0$, then $\cos 0 = 1$ so $u = 1$.

So,

$$V = -162\pi \int_0^1 (1-u^2)^3u^2(-du) = 162\pi \int_0^1 (1-3u^2+3u^4-u^6)u^2du =$$

$$= 162\pi \int_0^1 (u^2-3u^4+3u^6-u^8)du = 162\pi\left(\frac{u^3}{3} - 3\frac{u^5}{5} + 3\frac{u^7}{7} - \frac{u^9}{9}\right)\Bigg|_0^1 =$$

$$162\pi\left(\frac{1}{3} - \frac{3}{5} + \frac{3}{7} - \frac{1}{9}\right) = 162\pi\left(\frac{2}{9} + 3\left(\frac{1}{7} - \frac{1}{5}\right)\right) = 162\pi\left(\frac{2}{9} + 3\cdot\frac{-2}{35}\right) =$$

$$= 162\pi\left(\frac{70-54}{315}\right) = \frac{288\pi}{35}$$

Answer: $V = \frac{288\pi}{35}$.