Answer on Question #70926 – Math – Calculus

Question

How to find the double point of the curve y^2 = (x-2)^2 (x-1)?

Solution

The function $f(x,y)=0$ has a singular point P if in point P both first partial derivatives of the function are zero:

$$\frac{\partial}{\partial x}f(x,y)=\frac{\partial}{\partial y}f(x,y)=0\ .$$

Here

$$f(x,y)=y^2-(x-2)^2\cdot(x-1)=y^2-x^3+5x^2-8x+4$$

$$\frac{\partial}{\partial x}f(x,y)=-(x-2)^2-(x-1)(2x-4)=-3x^2+10x-8$$

$$\frac{\partial}{\partial y}f(x,y)=2y$$

$$\frac{\partial^2}{\partial x^2}f(x,y)=10-6x$$

$$\frac{\partial^2}{\partial y^2}f(x,y)=2$$

$$\frac{\partial^2}{\partial x\partial y}f(x,y)=0$$

To find a singular point, one should solve the system of equations

$$\begin{cases}\frac{\partial}{\partial x}f(x,y)=0\\ \frac{\partial}{\partial y}f(x,y)=0\end{cases}\Rightarrow\begin{cases}-(x-2)^2-(x-1)(2x-4)=0\\ 2y=0\end{cases}\Rightarrow\begin{cases}\left[\begin{array}{l}x=\frac{4}{3}\\ x=2\end{array}\right.\\ y=0\end{cases}\Rightarrow\left[\begin{array}{l}\begin{cases}x=\frac{4}{3}\\ y=0\end{cases}\\ \begin{cases}x=2\\ y=0\end{cases}\end{array}\right.$$

Point $\left(\frac{4}{3},0\right)$ does not lie on the curve $y^2=(x-2)^2(x-1)$ because $f\left(\frac{4}{3},0\right)\neq 0$.

Point (2,0) lies on the curve $y^2=(x-2)^2(x-1)$ because $f(2,0)=0$.

Point (2,0) is a double point because at least one of the second derivatives $\frac{\partial^2}{\partial x^2}f(x,y)\neq 0$, $\frac{\partial^2}{\partial y^2}f(x,y)\neq 0$ simultaneously for this point.

Type of the double point is determined by the expression

$$g(x,y) = \frac{\partial^2}{\partial x^2} f(x,y) \cdot \frac{\partial^2}{\partial y^2} f(x,y) - \left(\frac{\partial^2}{\partial x \partial y} f(x,y)\right)^2 = 2(10-6x) - 0^2 = 20 - 12x;$$

$$g(2,0) = 20 - 12 \times 2 < 0.$$

In this case, there are two branches of the curve passing through the point (2, 0). These branches have distinct tangents.

A slope m of the tangent can be found from the equation

$$\frac{\partial^2 f}{\partial y^2}(x_0, y_0) m^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) m + \frac{\partial^2 f}{\partial x^2}(x_0, y_0) = 0$$

$$2m^2 - 2 = 0$$

$$m = 1 \text{ or } m = -1$$

Curve $y^2 = (x-2)^2(x-1)$ can be rewritten as $y = (x-2)\sqrt{x-1}$ or

$y = -(x-2)\sqrt{x-1}$.

Such a point (2, 0) is called a node (a point of self-intersection).

Answer: (2, 0) is a double point, a point of self-intersection.