ANSWER on Question #70833 – Math – Geometry

QUESTION

Which of the following curves are regular?

1)

$$\gamma(t) = (\cos^2 t\,, \sin^2 t), \quad for\ -\infty < t < \infty$$

2)

$$\gamma(t) = (\cos^2 t\,, \sin^2 t), \quad for\ 0 < t < \frac{\pi}{2}$$

3)

$$\gamma(t) = (t, \cosh t), \quad for\ -\infty < t < \infty$$

Find unit speed reparametrisation of regular curve(s).

SOLUTION

By the definition, let $\gamma(t)$ be a curve. The velocity vector of $\gamma(t)$ at t is $\gamma'(t)$.

The speed at t is a length $|\gamma'(t)|$.

$$\gamma(t) = \big(x(t), y(t), z(t)\big) \to \gamma'(t) = \big(x'(t), y'(t), z'(t)\big)$$

$$|\gamma'(t)| = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}$$

By the definition, the curve $\gamma(t)$ is regular if all velocity vectors are different from zero, that is,

$$\gamma'(t) \neq 0, \forall t$$

In our case,

1)

$$\gamma(t) = (\cos^2 t\,, \sin^2 t), \quad for\ -\infty < t < \infty$$

$$x(t) = \cos^2 t \to x'(t) = 2\cdot(\cos t)\cdot(-\sin t) = -\sin(2t) \leftrightarrow \boxed{x'(t) = -\sin(2t)}$$

$$y(t) = \sin^2 t \to y'(t) = 2\cdot(\sin t)\cdot(\cos t) = \sin(2t) \leftrightarrow \boxed{y'(t) = \sin(2t)}$$

Then,

$$\gamma'(t) = (-\sin(2t)\,,\sin(2t))$$

$$\gamma'(\pi k) = \left(-\sin\left(2\cdot(\pi k)\right),\sin\left(2\cdot(\pi k)\right)\right) = (0{,}0)$$

Conclusion,

$$\gamma'(t) = 0, \forall t = \pi k, k \in \mathbb{Z} \to$$

$$\boxed{\gamma(t) = (\cos^2 t\,,\sin^2 t),\quad for\ -\infty < t < \infty\ isn't\ regular}$$

2)

$$\gamma(t) = (\cos^2 t\,,\sin^2 t),\quad for\ 0 < t < \frac{\pi}{2}$$

$$x(t) = \cos^2 t \to x'(t) = 2\cdot(\cos t)\cdot(-\sin t) = -\sin(2t) \leftrightarrow \boxed{x'(t) = -\sin(2t)}$$

$$y(t) = \sin^2 t \to y'(t) = 2\cdot(\sin t)\cdot(\cos t) = \sin(2t) \leftrightarrow \boxed{y'(t) = \sin(2t)}$$

Since the

$$0 < t < \frac{\pi}{2} \to 0 < (2t) < \pi$$

As we know

$$\sin(2t) \neq 0, \forall\ 0 < t < \frac{\pi}{2}$$

Then,

$$\gamma'(t) = (-\sin(2t)\,,\sin(2t)) \to \gamma'(t) \neq 0, \forall\ 0 < t < \frac{\pi}{2}$$

Conclusion,

$$\gamma'(t) \neq 0, \forall\ 0 < t < \frac{\pi}{2} \to$$

$$\boxed{\gamma(t) = (\cos^2 t\,,\sin^2 t),\quad for\ 0 < t < \frac{\pi}{2}\ is\ regular}$$

Now, we find the unit speed reparametrisation of regular curve $\gamma(t)$:

$$c = \int_0^{\frac{\pi}{2}} |\gamma'(t)|\, dt = \int_0^{\frac{\pi}{2}} \sqrt{(-\sin(2t))^2 + (\sin(2t))^2}\, dt = \int_0^{\frac{\pi}{2}} \sqrt{2} \cdot \sin(2t)\, dt$$

$$= \frac{\sqrt{2}}{2} \cdot (-\cos(2t))\Big|_0^{\frac{\pi}{2}} = -\frac{\sqrt{2}}{2} \cdot \left(\cos\left(2 \cdot \frac{\pi}{2}\right) - \cos(2 \cdot 0)\right) = -\frac{\sqrt{2}}{2} \cdot (\cos(\pi) - \cos(0)) =$$

$$= -\frac{\sqrt{2}}{2} \cdot (-1 - 1) = -\frac{\sqrt{2}}{2} \cdot (-2) = \sqrt{2} \to \boxed{c = \sqrt{2}}$$

Then

$$s(t) = \int_0^t c\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}t \to \boxed{t = \frac{s}{\sqrt{2}}}$$

Conclusion,

$$\boxed{\widetilde{\gamma(s)} = \gamma(t(s)) = \left(cos^2\left(\frac{s}{\sqrt{2}}\right), sin^2\left(\frac{s}{\sqrt{2}}\right)\right) \text{ } is\ the\ unit\ speed\ reparametrisation}$$

3)

$$\gamma(t) = (t, \cosh t), \quad for\ -\infty < t < \infty$$

$$x(t) = t \to x'(t) = 1 \leftrightarrow \boxed{x'(t) = 1}$$

$$y(t) = \cosh t \to y'(t) = \sinh t \leftrightarrow \boxed{y'(t) = \sinh t}$$

Then,

$$\gamma'(t) = (1, \sinh t)$$

Since the

$$1 \neq 0,$$

then

$$\gamma'(t) = (1, \sinh t) \neq 0, for -\infty < t < \infty \to$$

$$\boxed{\gamma(t) = (t, \cosh t), \quad for\ -\infty < t < \infty\ is\ regular}$$

Now, we find the unit speed reparametrisation of regular curve $\gamma(t)$

$$s(t) = \int_0^t |\gamma'(u)|\,du = \int_0^t \sqrt{1^2 + (\sinh u)^2}\,du = \begin{bmatrix} As\ we\ know \\ \cosh^2 x - \sinh^2 x = 1 \\ \cosh^2 x = 1 + \sinh^2 x \end{bmatrix} =$$

$$= \int_0^t \sqrt{\cosh^2 u}\,du = \int_0^t \cosh u\,du = \sinh u|_0^t = \sinh t$$

$$s(t) = \sinh t \rightarrow \boxed{t(s) = \operatorname{asinh} s}$$

Conclusion,

$$\boxed{\widetilde{\gamma(s)} = \gamma\big(t(s)\big) = (\operatorname{asinh} s\,, \cosh(\operatorname{asinh} s)) - the\ unit\ speed\ reparametrisation}$$

ANSWER:

1)

$$\gamma(t) = (\cos^2 t\,, \sin^2 t) \quad for\ -\infty < t < \infty$$

$$isn't\ regular$$

2)

$$\gamma(t) = (\cos^2 t\,, \sin^2 t), \quad for\ 0 < t < \frac{\pi}{2}$$

$$is\ regular$$

$$\widetilde{\gamma(s)} = \left(cos^2\left(\frac{s}{\sqrt{2}}\right), sin^2\left(\frac{s}{\sqrt{2}}\right)\right)\ is\ the\ unit\ speed\ reparametrisation$$

3)

$$\gamma(t) = (t, \cosh t) \quad for\ -\infty < t < \infty$$

$$is\ regular$$

$$\widetilde{\gamma(s)} = (\operatorname{asinh} s\,, \cosh(\operatorname{asinh} s))\ is\ the\ unit\ speed\ reparametrisation$$