ANSWER on Question #70832 – Math – Geometry

QUESTION

Show that the following curves are unit–speed:

1)

$$s(t) = \left(\frac{1}{3}(1+t)^{\frac{3}{2}}, \quad \frac{1}{3}(1-t)^{\frac{3}{2}}, \quad \frac{t}{\sqrt{2}}\right)$$

2)

$$s(t) = \left(\frac{4}{5}\cos t, \ 1-\sin t, \quad -\frac{3}{5}\sin t\right)$$

SOLUTION

By the definition, let $s(t)$ be a curve. The velocity vector of $s(t)$ at t is $s'(t)$.

The speed at t is a length $|s'(t)|$. The unit-speed is $|s'(t)| = 1, t \geq 0$

$$s(t) = \big(x(t), y(t), z(t)\big) \to s'(t) = \big(x'(t), y'(t), z'(t)\big)$$

$$|s'(t)| = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}$$

In our case,

1)

$$s(t) = \left(\frac{1}{3}(1+t)^{\frac{3}{2}}, \quad \frac{1}{3}(1-t)^{\frac{3}{2}}, \quad \frac{t}{\sqrt{2}}\right)$$

Then,

$$x(t) = \frac{1}{3}(1+t)^{\frac{3}{2}} \to x'(t) = \frac{1}{3} \cdot \frac{3}{2} \cdot (1+t)^{\frac{3}{2}-1} \cdot 1 = \frac{1}{2} \cdot (1+t)^{\frac{1}{2}}$$

$$\boxed{x'(t) = \frac{1}{2} \cdot (1+t)^{\frac{1}{2}}}$$

$$y(t) = \frac{1}{3}(1-t)^{\frac{3}{2}} \to y'(t) = \frac{1}{3} \cdot \frac{3}{2} \cdot (1-t)^{\frac{3}{2}-1} \cdot (-1) = -\frac{1}{2} \cdot (1-t)^{\frac{1}{2}}$$

$$\boxed{y'(t) = -\frac{1}{2} \cdot (1-t)^{\frac{1}{2}}}$$

$$z(t)=\frac{t}{\sqrt{2}}\rightarrow \boxed{z'(t)=\frac{1}{\sqrt{2}}}$$

Then,

$$|s'(t)|=\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2+\left(z'(t)\right)^2}=$$

$$=\sqrt{\left(\frac{1}{2}\cdot(1+t)^{\frac{1}{2}}\right)^2+\left(-\frac{1}{2}\cdot(1-t)^{\frac{1}{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}=$$

$$=\sqrt{\frac{(1+t)^{2\cdot\frac{1}{2}}}{4}+\frac{(1-t)^{2\cdot\frac{1}{2}}}{4}+\frac{1}{2}}=\sqrt{\frac{1+t}{4}+\frac{1-t}{4}+\frac{1\cdot 2}{2\cdot 2}}=\sqrt{\frac{1+t+1-t+2}{4}}=\sqrt{\frac{4}{4}}=1$$

Conclusion,

$$\boxed{|s'(t)|=1,\forall t\geq 0}$$

2)

$$s(t)=\left(\frac{4}{5}\cos t\,,\ \ 1-\sin t\,,\qquad -\frac{3}{5}\cos t\right)$$

Then,

$$x(t)=\frac{4}{5}\cos t\rightarrow \boxed{x'(t)=-\frac{4}{5}\sin t}$$

$$y(t)=\ 1-\sin t\rightarrow \boxed{y'(t)=-\cos t}$$

$$z(t)=-\frac{3}{5}\cos t\rightarrow \boxed{z'(t)=\frac{3}{5}\sin t}$$

Then,

$$|s'(t)|=\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2+\left(z'(t)\right)^2}=$$

$$=\sqrt{\left(-\frac{4}{5}\sin t\right)^2+(-\cos t)^2+\left(\frac{3}{5}\sin t\right)^2}=\sqrt{\frac{16\sin^2 t}{25}+\frac{9\sin^2 t}{25}+\cos^2 t}=$$

$$= \sqrt{\frac{(16+9)\sin^2 t}{25} + \cos^2 t} = \sqrt{\frac{25\sin^2 t}{25} + \cos^2 t} = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1$$

Conclusion,

$$\boxed{|s'(t)| = 1, \forall t \geq 0}$$

ANSWER:

1)

$$s(t) = \left(\frac{1}{3}(1+t)^{\frac{3}{2}}, \quad \frac{1}{3}(1-t)^{\frac{3}{2}}, \quad \frac{t}{\sqrt{2}}\right) \to |s'(t)| = 1, \forall t \geq 0$$

2)

$$s(t) = \left(\frac{4}{5}\cos t, \ 1 - \sin t, \quad -\frac{3}{5}\sin t\right) \to |s'(t)| = 1, \forall t \geq 0$$