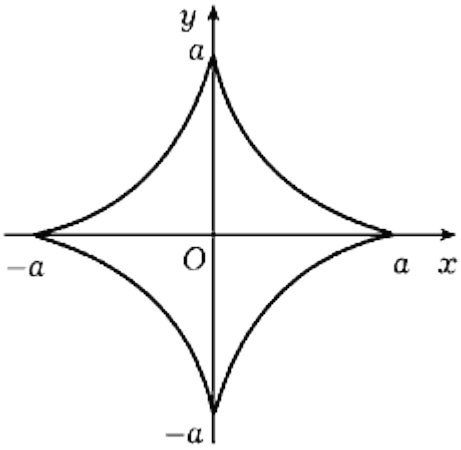
Answer on Question #70824 – Math – Geometry

Question

1. Find the length of astroid $x = a\cos^3 t, y = a\sin^3 t, [0, 2\pi]$

Solution



In view of the symmetry of the curve, it's enough to find one-fourth l of the length of the arc for arstroid (L is the length of astroid, $L = 4l$), the parameter changes from 0 to $\frac{\pi}{2}$.

Find the differentials:

$$dx = -3a\cos^2 t \sin t\,;\ dy = 3a\sin^2 t \cos t\,;$$

Hence we find

$$dx = \sqrt{(dx)^2 + (dy)^2} = \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t} =$$
$$= \sqrt{9a^2\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)} = \sqrt{9a^2\cos^2 t\sin^2 t} = 3a\cos t \sin t$$
$$= \frac{3}{2}a \sin 2t$$

Integrating the resulting expression for dx in the range from 0 to $\frac{\pi}{2}$, we get

$$l = \int_0^{\pi/2} \frac{3}{2}a \sin 2t\, dt = \frac{3}{2}a \int_0^{\pi/2} \sin 2t\, dt = -\frac{3}{4}a \cos 2t \Big|_0^{\frac{\pi}{2}} = \frac{3}{4}a + \frac{3}{4}a = \frac{3}{2}a$$

$$\Rightarrow L = 4 \times \frac{3}{2}a = 6a$$

Answer: the length of the astroid is $6a$.