Answer on Question #70612, Math / Differential Equations

The normal form of the differential equation

$$y'' - xy' + (4x^2 - 1)y = 0$$

Solution

Consider a second order linear ODE in the standard form

$$y'' + p(x)y' + q(x)y = 0 \quad (1)$$

By a change of dependent variable, (1) can be written as

$$u'' + Q(x)u = 0 \quad (2)$$

which is called the normal form of (1).

To find the transformation, let use put $y(x) = \mathrm{u(x)v(x)}$.

When this is substituted in (1), we get

$$y' = u'v + v'u$$

$$y'' = u''v + u'v' + v''u + v'u'$$

$$vu'' + (2v' + pv)u' + (v'' + pv' + qv)u = 0$$

Now we set the coefficient of u' to zero. This gives

$$2v' + pv = 0 => \frac{dv}{v} = -\frac{p}{2}dx => v = e^{-\int p/2dx}$$

Now coefficient of u becomes

$$\left(q - \frac{1}{4}p^2 - \frac{1}{2}p'\right)v = Q(x)v$$

Since v is nonzero, cancelling v we get the required normal form. Also, since v never vanishes, u vanishes if and only if y vanishes. Thus, the above transformation has no effect on the zeros of solution.

$$y'' - xy' + (4x^2 - 1)y = 0$$

We have that

$$p(x) = -x, q(x) = 4x^2 - 1$$

Then

$$v = e^{-\int(-x)/2dx} = e^{x^2/4}$$

Now

$$Q(x) = 4x^2 - 1 - \frac{1}{4}(-x)^2 - \frac{1}{2}(-x)'$$

$$Q(x) = 4x^2 - 1 - \frac{1}{4}x^2 + \frac{1}{2}$$

$$Q(x) = \frac{15}{4}x^2 - \frac{1}{2}$$

Thus, the equation in normal form becomes

$$u'' + \left(\frac{15}{4}x^2 - \frac{1}{2}\right)u = 0,$$

where $u = ye^{-x^2/4}$