Answer on Question #70267 – Math – Differential Equations

Question

Reduce the following PDE to a set of three ODEs by the method of separation of variables.

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}=0$$

Solution

Represent $u=u(r,\theta,\phi)$ as a product of two functions depending on r and $(\theta,\ \varphi)$:

$$u(r,\theta,\phi)=R(r)Y(\theta,\varphi)$$

Substituting this expression into original equation we obtain

$$Y\frac{d^2R}{dr^2}+\frac{2Y}{r}\frac{dR}{dr}+\frac{R}{r^2}\frac{\partial^2 Y}{\partial \theta^2}+\frac{R\cot\theta}{r^2}\frac{\partial Y}{\partial \theta}+\frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2}=0$$

Multiplying this equation by $\frac{r^2}{RY}$ we get

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{2r}{R}\frac{dR}{dr}+\frac{1}{Y}\frac{\partial^2 Y}{\partial \theta^2}+\frac{\cot\theta}{Y}\frac{\partial Y}{\partial \theta}+\frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2}=0$$

or

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{2r}{R}\frac{dR}{dr}=-\left(\frac{1}{Y}\frac{\partial^2 Y}{\partial \theta^2}+\frac{\cot\theta}{Y}\frac{\partial Y}{\partial \theta}+\frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2}\right)$$

Equating both sides of the equality to a some constant λ, we obtain two equations.

First equation for $R(r)$:

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{2r}{R}\frac{dR}{dr}=\lambda$$

or

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-\lambda R=0$$

Second equation for $Y(\theta,\varphi)$:

$$-\left(\frac{1}{Y}\frac{\partial^2 Y}{\partial \theta^2}+\frac{\cot\theta}{Y}\frac{\partial Y}{\partial \theta}+\frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2}\right)=\lambda$$

or

$$\frac{\partial^2 Y}{\partial \theta^2}+\cot\theta\frac{\partial Y}{\partial \theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2}+\lambda Y=0$$

Now we represent $Y(\theta,\varphi)$ as a product of two functions depending on θ and φ

$$Y(\theta,\varphi)=V(\theta)\Phi(\varphi)$$

and substitute it into the last equation

$$\Phi\frac{d^2V}{d\theta^2}+\Phi\cot\theta\frac{dV}{d\theta}+\frac{V}{\sin^2\theta}\frac{d^2\Phi}{d\varphi^2}+\lambda V\Phi=0$$

Multiplying this equation by $\frac{\sin^2\theta}{\Phi V}$ we get

$$\frac{\sin^2\theta}{V}\frac{d^2V}{d\theta^2}+\frac{\cos\theta\,\sin\theta}{V}\frac{dV}{d\theta}+\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}+\lambda\sin^2\theta=0$$

or

$$\frac{\sin^2\theta}{V}\frac{d^2V}{d\theta^2}+\frac{\cos\theta\,\sin\theta}{V}\frac{dV}{d\theta}+\lambda\sin^2\theta=-\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}$$

Equating both sides of this equality to a some constant m^2, we obtain two equations.

First equation for $V(\theta)$:

$$\frac{\sin^2\theta}{V}\frac{d^2V}{d\theta^2}+\frac{\cos\theta\,\sin\theta}{V}\frac{dV}{d\theta}+\lambda\sin^2\theta=m^2$$

or

$$\frac{d^2V}{d\theta^2}+\cot\theta\frac{dV}{d\theta}+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)V=0$$

The second equation for $\Phi(\varphi)$:

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=m^2$$

or

$$\frac{d^2\Phi}{d\varphi^2}+m^2\Phi=0$$

Thus representing the original function $u(r,\theta,\varphi)$ as a product

$$u(r,\theta,\phi)=R(r)Y(\theta,\varphi)=R(r)V(\theta)\Phi(\varphi)$$

we obtained three equations for the functions $R(r)$, $V(\theta)$ and $\Phi(\varphi)$:

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-\lambda R=0$$

$$\frac{d^2V}{d\theta^2}+\cot\theta\frac{dV}{d\theta}+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)V=0$$

$$\frac{d^2\Phi}{d\varphi^2}+m^2\Phi=0$$

where λ and m are the some constants

Answer: the original PDE is reduced to a set of three ODEs

$$r^2\frac{d^2R(r)}{dr^2}+2r\frac{dR(r)}{dr}-\lambda R(r)=0$$

$$\frac{d^2V(\theta)}{d\theta^2} + \cot\theta \frac{dV(\theta)}{d\theta} + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)V(\theta) = 0$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} + m^2\Phi(\varphi) = 0$$