Answer on Question #70193 – Math – Functional Analysis

Question

Show that the absolute value of a linear functional has the subadditive and positive homogeneous properties.

Solution

Let $f : X \to \mathrm{R}$ be a linear functional and X a vector space. We show that $|f|$ has the subadditive and positive homogeneous properties.

Consider arbitrary $x, y \in X$. Since f is a linear functional, f is a additive function. Thus, $f(x+y) = f(x) + f(y)$. Then

$$|f(x+y)| = |f(x) + f(y)| \le |f(x)| + |f(x)|.$$

So, $|f(x+y)| \le |f(x)| + |f(x)|$ for each $x, y \in X$. This means that f has the subadditive property.

Consider any $x \in X$ and $\alpha \in \mathrm{R}$, $\alpha > 0$. Since f is a linear functional, f is homogeneous. Thus, $f(\alpha x) = \alpha f(x)$. Then

$$|f(\alpha x)| = |\alpha f(x)| = |\alpha||f(x)| = \alpha |f(x)|$$

because $\alpha > 0$. So, $|f(\alpha x)| = \alpha |f(x)|$ for each $x \in X$ and $\alpha \in \mathrm{R}$, $\alpha > 0$. This means that f has the positive homogeneous property.