

Answer on Question #70105 - Math – Calculus

Question

Determine the values of a and b for which the function

$$f(x) = \begin{cases} (\sin 3x + a \sin 2x + b \sin x)/x^5, & x \neq 0; \\ 1, & x = 0. \end{cases}$$

is continuous at $x=0$.

Solution

Function $f(x)$ is continuous at $x = x_0$ if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

So we find a and b assuming that the condition

$$\lim_{x \rightarrow 0} \frac{\sin 3x + a \sin 2x + b \sin x}{x^5} = 1$$

hold.

Consider the expression $B = \sin 3x + a \sin 2x + b \sin x$

Use the formulas

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

$$\sin 2x = 2 \sin x \cos x$$

Then we get

$$B = 3 \sin x - 4 \sin^3 x + 2a \sin x \cos x + b \sin x$$

or

$$B = (3 + b) \sin x + 2 \sin x (a \cos x - 2 \sin^2 x)$$

Now for the second term we use the formulas

$$\cos x = 1 - 2 \sin^2 \frac{x}{2}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

Then we get

$$B = (3 + b) \sin x + 2 \sin x \left(a - 2a \sin^2 \frac{x}{2} - 8 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2} \right)$$

or

$$B = (3 + b + 2a) \sin x + 2 \sin x \left(-2a \sin^2 \frac{x}{2} - 8 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2} \right)$$

Now we use $\cos^2 \frac{x}{2} = 1 - \sin^2 \frac{x}{2}$

$$B = (3 + b + 2a) \sin x + 2 \sin x \left(-2a \sin^2 \frac{x}{2} - 8 \sin^2 \frac{x}{2} + 8 \sin^4 \frac{x}{2} \right)$$

or

$$B = (3 + b + 2a)\sin x - 4\sin^3 x(a + 4) + 16\sin x \sin^4 \frac{x}{2}$$

Substitute B into limit

$$\lim_{x \rightarrow 0} \frac{(3 + b + 2a)\sin x - 4\sin^3 x(a + 4) + 16\sin x \sin^4 \frac{x}{2}}{x^5}$$

We choose a and b from the condition that the first and second terms vanish, i.e.

$$3 + b + 2a = 0; \quad a + 4 = 0.$$

We get $a = -4$, $b = 5$

and the limit becomes

$$\lim_{x \rightarrow 0} \frac{16\sin x \sin^4 \frac{x}{2}}{x^5} = 16 \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin^4 \frac{x}{2}}{x^4} = 16 \cdot 1 \cdot \frac{1}{16} = 1$$

Thus at $a = -4$, $b = 5$ we have $\lim_{x \rightarrow 0} f(x) = f(0) = 1$ and the function $f(x)$ is continuous at $x = 0$.

Answer: the function

$$f(x) = \begin{cases} (\sin 3x + a \sin 2x + b \sin x)/x^5, & x \neq 0; \\ 1, & x = 0. \end{cases}$$

is continuous at $x=0$ when $a = -4$, $b = 5$.

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