Answer on Question #70103 – Math – Real Analysis

Question

Evaluate $\iiint_S \; z^2 dxdydz$ where S is the solid region between the spheres $\rho = 1$ and $\rho = 2$, by using spherical coordinates.

Solution

Let us use the spherical coordinates

(see http://mathworld.wolfram.com/SphericalCoordinates.html):

$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$, $r \in [1, 2], \theta \in [0, 2\pi), \varphi \in [0, \pi]$. The Jacobian is $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2 \sin\varphi$. Then the original integral is equal to $\iiint_D \; r^4(\cos\varphi)^2 \sin\varphi \, dr d\theta d\varphi =$

$\int_1^2 \int_0^\pi \int_0^{2\pi} r^4(\cos\varphi)^2 \sin\varphi \, d\theta d\varphi dr = \left(\int_1^2 r^4 dr\right)\left(\int_0^\pi (\cos\varphi)^2 \sin\varphi \, d\varphi\right)\left(\int_0^{2\pi} d\theta\right) =$

$= 2\pi\left(\frac{r^5}{5}\Big|_{r=1}^2\right)\left(-\int_0^\pi (\cos\varphi)^2 d(\cos\varphi)\right) = -\frac{62\pi}{5}\left(\frac{(\cos\varphi)^3}{3}\Big|_{\varphi=0}^\pi\right) = \frac{124\pi}{15} = 8\frac{4}{15}\pi.$

Answer: $8\frac{4}{15}\pi$.