Answer on Question #70004 – Math – Linear Algebra

Question

For any four vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ determine:

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})+(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})$$

Solution

Suppose

$\vec{a}=a_1\vec{\imath}+a_2\vec{\jmath}+a_3\vec{k}$

$\vec{b}=b_1\vec{\imath}+b_2\vec{\jmath}+b_3\vec{k}$

$\vec{c}=c_1\vec{\imath}+c_2\vec{\jmath}+c_3\vec{k}$

$\vec{d}=d_1\vec{\imath}+d_2\vec{\jmath}+d_3\vec{k}$

Then

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=(a_2b_3-a_3b_2)\vec{\imath}+(a_3b_1-a_1b_3)\vec{\jmath}+(a_1b_2-a_2b_1)\vec{k}$$

$$\vec{c}\times\vec{d}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ c_1 & c_2 & c_3\\ d_1 & d_2 & d_3\end{vmatrix}=(c_2d_3-c_3d_2)\vec{\imath}+(c_3d_1-c_1d_3)\vec{\jmath}+(c_1d_2-c_2d_1)\vec{k}$$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=(b_2c_3-b_3c_2)\vec{\imath}+(b_3c_1-b_1c_3)\vec{\jmath}+(b_1c_2-b_2c_1)\vec{k}$$

$$\vec{a}\times\vec{d}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ a_1 & a_2 & a_3\\ d_1 & d_2 & d_3\end{vmatrix}=(a_2d_3-a_3d_2)\vec{\imath}+(a_3d_1-a_1d_3)\vec{\jmath}+(a_1d_2-a_2d_1)\vec{k}$$

$$\vec{c}\times\vec{a}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix}=(c_2a_3-c_3a_2)\vec{\imath}+(c_3a_1-c_1a_3)\vec{\jmath}+(c_1a_2-c_2a_1)\vec{k}$$

$$\vec{b}\times\vec{d}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ b_1 & b_2 & b_3\\ d_1 & d_2 & d_3\end{vmatrix}=(b_2d_3-b_3d_2)\vec{\imath}+(b_3d_1-b_1d_3)\vec{\jmath}+(b_1d_2-b_2d_1)\vec{k}$$

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=(a_2b_3-a_3b_2)(c_2d_3-c_3d_2)+$$

$$+(a_3b_1-a_1b_3)(c_3d_1-c_1d_3)+(a_1b_2-a_2b_1)(c_1d_2-c_2d_1)=$$

$$=a_2b_3c_2d_3-a_3b_2c_2d_3-a_2b_3c_3d_2+a_3b_2c_3d_2+a_3b_1c_3d_1-a_1b_3c_3d_1-$$

$-a_3b_1c_1d_3+a_1b_3c_1d_3+a_1b_2c_1d_2-a_2b_1c_1d_2-a_1b_2c_2d_1+a_2b_1c_2d_1$

$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})=(b_2c_3-b_3c_2)(a_2d_3-a_3d_2)+$

$+(b_3c_1-b_1c_3)(a_3d_1-a_1d_3)+(b_1c_2-b_2c_1)(a_1d_2-a_2d_1)=$

$=b_2c_3a_2d_3-b_3c_2a_2d_3-b_2c_3a_3d_2+b_3c_2a_3d_2+b_3c_1a_3d_1-b_1c_3a_3d_1-$

$-b_3c_1a_1d_3+b_1c_3a_1d_3+b_1c_2a_1d_2-b_2c_1a_1d_2-b_1c_2a_2d_1+b_2c_1a_2d_1$

$(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})=(c_2a_3-c_3a_2)(b_2d_3-b_3d_2)+$

$+(c_3a_1-c_1a_3)(b_3d_1-b_1d_3)+(c_1a_2-c_2a_1)(b_1d_2-b_2d_1)=$

$=c_2a_3b_2d_3-c_3a_2b_2d_3-c_2a_3b_3d_2+c_3a_2b_3d_2+c_3a_1b_3d_1-c_1a_3b_3d_1-$

$-c_3a_1b_1d_3+c_1a_3b_1d_3+c_1a_2b_1d_2-c_2a_1b_1d_2-c_1a_2b_2d_1+c_2a_1b_2d_1$

Hence

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})+(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})=$

$=a_2b_3c_2d_3-a_3b_2c_2d_3-a_2b_3c_3d_2+a_3b_2c_3d_2+a_3b_1c_3d_1-a_1b_3c_3d_1-$

$-a_3b_1c_1d_3+a_1b_3c_1d_3+a_1b_2c_1d_2-a_2b_1c_1d_2-a_1b_2c_2d_1+a_2b_1c_2d_1+$

$+b_2c_3a_2d_3-b_3c_2a_2d_3-b_2c_3a_3d_2+b_3c_2a_3d_2+b_3c_1a_3d_1-b_1c_3a_3d_1-$

$-b_3c_1a_1d_3+b_1c_3a_1d_3+b_1c_2a_1d_2-b_2c_1a_1d_2-b_1c_2a_2d_1+b_2c_1a_2d_1+$

$+c_2a_3b_2d_3-c_3a_2b_2d_3-c_2a_3b_3d_2+c_3a_2b_3d_2+c_3a_1b_3d_1-c_1a_3b_3d_1-$

$-c_3a_1b_1d_3+c_1a_3b_1d_3+c_1a_2b_1d_2-c_2a_1b_1d_2-c_1a_2b_2d_1+c_2a_1b_2d_1=$

$=a_1(-b_3c_3d_1+b_3c_1d_3+b_2c_1d_2-b_2c_2d_1-b_3c_1d_3+b_1c_3d_3+b_1c_2d_2-$

$-b_2c_1d_2+c_3b_3d_1-c_3b_1d_3-c_2b_1d_2+c_2b_2d_1)+a_2(b_3c_2d_3-b_3c_3d_2-$

$-b_1c_1d_2+b_1c_2d_1+b_2c_3d_3-b_3c_2d_3-b_1c_2d_1+b_2c_1d_1-c_3b_2d_3+$

$+c_3b_3d_2+c_1b_1d_2-c_1b_2d_1)+a_3(-b_2c_2d_3+b_2c_3d_2+b_1c_3d_1-$

$-b_1c_1d_3-b_2c_3d_2+b_3c_2d_2+b_3c_1d_1-b_1c_3d_1+c_2b_2d_3-c_2b_3d_2-$

$-c_1b_3d_1+c_1b_1d_3)=0$

Answer: $\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)+\left(\vec{b}\times\vec{c}\right)\cdot\left(\vec{a}\times\vec{d}\right)+(\vec{c}\times\vec{a})\cdot\left(\vec{b}\times\vec{d}\right)=0.$

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