Answer on Question #69951 – Math – Discrete Mathematics

Question

Find the number of integer solutions to the equation $x_1 + x_2 + x_3 = 40$ where

$6 \le x_1 \le 15,\ \ 5 \le x_2 \le 20$ and $10 \le x_3 \le 25$.

Solution

Method 1

Let $x_1, x_2, \ldots, x_m$ be integers. Then the number of solutions to the equation

$$x_1 + x_2 + \cdots + x_m = n$$

subject to the conditions $a_1 \le x_1 \le b_1,\ \ a_2 \le x_2 \le b_2,\ \ldots,\ a_m \le x_m \le b_m$

is equal to the coefficient of x^n in

$(x^{a_1} + x^{a_1+1} + \cdots + x^{b_1}) \cdot (x^{a_2} + x^{a_2+1} + \cdots + x^{b_2}) \cdots (x^{a_m} + x^{a_m+1} + \cdots + x^{b_m})$.

(see http://www.careerbless.com/aptitude/qa/permutations_combinations_imp8.php)

Applying this theorem to this problem one gets that the number of solutions to the equation

$$x_1 + x_2 + x_3 = 40$$

subject to conditions

$6 = a_1 \le x_1 \le b_1 = 15,\ \ 5 = a_2 \le x_2 \le b_2 = 20,\ \ldots,\ 10 = a_3 \le x_3 \le b_3 = 25$

is equal to the coefficient of x^{40} in

$$(x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15}) \cdot$$
$$(x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19} + x^{20}) \cdot$$
$$(x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19} + x^{20} + x^{21} + x^{22} + x^{23} + x^{24} + x^{25}) =$$
$$= x^6 x^5 x^{10} (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9) \cdot$$
$$\cdot (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15}) \cdot$$
$$\cdot (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15}) =$$
$$= x^{21} (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9) \cdot$$
$$\cdot (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15}) \cdot$$
$$\cdot (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15})$$

Thus, the number of integer solutions is 135.

Method 2

From the equation $x_1 + x_2 + x_3 = 40$ we can express one of variables in terms of others: $x_3 = 40 - (x_1 + x_2)$. Thus, $10 \le 40 - (x_1 + x_2) \le 25$, or, $-30 \le -(x_1 + x_2) \le -15$, or $15 \le x_1 + x_2 \le 30$.

For each integer from 15 to 30 let's count the number of pairs of integers (x_1, x_2) such that $6 \le x_1 \le 15$, $5 \le x_2 \le 20$.

Let $x_1 + x_2 = n$, where $15 \le n \le 30$. The minimal value of x_1 (even less than 6) for which there exists x_2 such that $x_1 + x_2 = n$ is $n - 20$, i.e. n minus upper limit of x_2. Taking into account the lower limit of x_1 being 6, the minimal value of x_1 really is $\max\{(n-20), 6\}$. Similarly, the maximal value of x_2 (even greater than 15) for which there exists appropriate x_2 is $n - 5$, i.e. n minus lower limit of x_2. Taking into account the upper limit of x_1 being 15, the maximal value of x_1 really is $\min\{(n-5), 15\}$. Then the number of pairs of integers (x_1, x_2) such that that $6 \le x_1 \le 15$, $5 \le x_2 \le 20$ and $x_1 + x_2 = n$ is the number of integers between $a_n = \max\{(n-20), 6\}$ and $b_n = \min\{(n-5), 15\}$, i.e. $(b_n - a_n + 1)$. In the table below columns correspond to value of n while blue cells in each column correspond to valid values of x_1:

	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
6																
7																
8																
9																
10																
11																
12																
13																
14																
15																

For $15 \le n \le 26$, $a_n = 6$ and for the rest, $27 \le n \le 30$, $a_n = n - 20$. For $15 \le n \le 19$, $b_n = n - 5$, and for the rest, $20 \le n \le 30$, $b_n = 15$. Dividing n into three groups: below 20 (A), above 26 (B) and between (C):

$$A = \sum_{n=15}^{19} (b_n - a_n + 1) = \sum_{n=15}^{19} \big((n-5) - 6 + 1\big) = \sum_{n=15}^{19} (n - 10) = \sum_{n=5}^{9} n = (9 - 5 + 1)\frac{9+5}{2}$$
$$= 5 \times 7 = 35.$$

$$B = \sum_{n=20}^{26} (b_n - a_n + 1) = \sum_{n=20}^{26} (15 - 6 + 1) = \sum_{n=20}^{26} 10 = 10(26 - 20 + 1) = 10 \times 7 = 70.$$

$$C = \sum_{n=27}^{30} (b_n - a_n + 1) = \sum_{n=27}^{30} (15 - (n-20) + 1) = \sum_{n=27}^{30} (36 - n) = \sum_{n=36-27}^{36-30} n = \sum_{n=9}^{6} n$$
$$= \sum_{n=6}^{9} n = (9 - 6 + 1)\frac{6+9}{2} = 4 \times 7.5 = 30.$$

$$N = A + B + C = 35 + 70 + 30 = 135.$$

Answer: 135.