Answer on Question #69784 – Math – Differential Geometry

Question

Determine the square of the arc element for the curvilinear coordinate system (u, v, w) whose coordinates are related to the Cartesian coordinates as follows:

$x = 3u + v - w; y = u + 2v + 2w; z = 2u - v - w.$

Solution

For any orthogonal curvilinear coordinates q_i $(i = 1..3)$ the square of the line element is given by

$$ds^2 = h_1^2(dq_1)^2 + h_2^2(dq_2)^2 + h_3^2(dq_3)^2,$$

where $h_i = \left|\frac{\partial \mathbf{r}}{\partial q_i}\right|$.

Thus

$$h_1 = \left|\frac{\partial \mathbf{r}}{\partial q_1}\right| = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2} = \sqrt{9 + 1 + 4} = \sqrt{14},$$

$$h_2 = \left|\frac{\partial \mathbf{r}}{\partial q_2}\right| = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2} = \sqrt{1 + 4 + 1} = \sqrt{6},$$

$$h_3 = \left|\frac{\partial \mathbf{r}}{\partial q_3}\right| = \sqrt{\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2} = \sqrt{4 + 1 + 1} = \sqrt{6}.$$

Finally the square of the arc element

$$ds^2 = 14(du)^2 + 6(dv)^2 + 6(dw)^2$$

Answer: $ds^2 = 14(du)^2 + 6(dv)^2 + 6(dw)^2$.