Answer on Question #69326 – Math – Algebra

Question

$x^3 + y^3 + z^3 – 3xyz = (x + y + z)(x + wy + w^2z)(x + w^2y + wz)$

Solution

Consider

$(x+y+z)[x^2+y^2+z^2-(xy+yz+zx)] =$
$= x^3+ x y^2+ x z^2- x^2 y- x yz - x^2 z +$
$+x^2 y + y^3 + y z^2 - x y^2 - y^2z- xyz+ x^2z+ y^2 z + z^3 - xyz- yz^2 - x z^2 =$
$= x^3+ y^3+ z^3+[x y^2- x y^2 +x z^2- x z^2- x^2 y+x^2 y- x^2 z+ x^2z+ y z^2- yz^2-$
$- y^2z+ y^2 z]- 3xyz= x^3+ y^3+ z^3- 3xyz$

It was proved that
$(x+y+z)[x^2+y^2+z^2-(xy+yz+zx)] = x^3+ y^3+ z^3- 3xyz$ (1)

Consider

$(x+y+z)[x^2+y^2+z^2-(xy+yz+zx)] =$
$=(x+y+z)[x^2+y^2+z^2+(w+w^2)(xy+yz+zx)]$ (since $w+w^2= -1$) =
$=(x+y+z)[x^2+y^2+z^2+xyw+yzw+zxw+xyw^2+ yzw^2 + zxw^2] =$
$=(x+y+z)[x^2+y^2+z^2+xwy+wyz+xwz+xyw^2+ yzw^2 + zxw^2] =$
$=(x+y+z)[x^2+w^3y^2+w^3z^2+xwy+w^4yz+ xwz+xyw^2+ yzw^2 + zxw^2] =$

(since $w^3=1$, $w^4=w$)
$=(x+y+z)[x^2+xyw^2+xwz+wyx+w^3y^2+w^2yz+w^2zx+ w^4zy+ w^3z^2] =$
$=(x+y+z)[x(x+w^2y+wz)+wy(x+w^2y+wz)+w^2z(x+w^2y+wz)] =$
$=(x+y+z)(x+ wy + w^2z)(x + w^2y +wz)$

It was proved that
$(x+y+z)[x^2+y^2+z^2-(xy+yz+zx)] =(x+y+z)(x+ wy + w^2z)(x + w^2y +wz)$ (2)

It follows from (1) and (2) that

$x^3 + y^3 + z^3 – 3xyz = (x + y + z)(x + wy + w^2z)(x + w^2y + wz)$
Q.E.D.