Answer on Question #69114 – Math – Real Analysis

Question

Show that the sequence $f_n(x) = \frac{x}{1+2nx^2}, x \in [1, \infty)$ is uniformly convergent in $[1, \infty)$.

Proof

Obviously the pointwise limit of $f_n(x)$ is 0. Indeed, for all fixed $x_0 \in [1, \infty)$

$$\lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty} \frac{x_0}{1+2nx_0^2} = x_0 \lim_{n\to\infty} \frac{1}{1+2nx_0^2} = \{1 + 2nx_0^2 \to \infty \; as \; n \to \infty\} = 0.$$

Now we apply the definition of the uniform convergence (see https://en.wikipedia.org/wiki/Uniform_convergence#Definition).

In our case the limit function is $f(x) \equiv 0$, and now we must find

$$a_n = \sup_{x\in[1,\infty)} |f_n(x) - f(x)| = \sup_{x\in[1,\infty)} \frac{x}{1+2nx^2} > 0.$$

Applying a derivative we obtain:

$$\frac{df_n(x)}{dx} = \frac{1+2nx^2-4nx^2}{(1+2nx^2)^2} = \frac{1-2nx^2}{(1+2nx^2)^2} < 0, \text{ where } n \in \mathbb{N}, x \in [1, \infty).$$

Then $f_n(x) = \frac{x}{1+2nx^2}$ monotonically decreases in x when $x \in [1, \infty)$, and

$$\sup_{x\in[1,\infty)} f_n(x) = \sup_{x\in[1,\infty)} \frac{x}{1+2nx^2} = f_n(1) = \frac{1}{1+2n}.$$

Since $a_n = \frac{1}{1+2n} \to 0$ as $n \to \infty$ (because $1 + 2n \to \infty$ as $n \to \infty$), we conclude that

$f_n(x) = \frac{x}{1+2nx^2}$ is uniformly convergent in $[1, \infty)$.