Answer on Question #69111 – Math – Real Analysis

Question

Show that the function $f: [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 2, & x \notin \mathbb{Q} \end{cases}$ is not Riemann integrable.

Proof

The definition of Riemann integral is here:

https://en.wikipedia.org/wiki/Riemann_integral#Definition .

Let us take an arbitrary partition of interval $[0,1]$: $0 = x_0 < x_1 < \cdots < x_n = 1$. Now we shall take the tagged partition $P(x,t)$ of $[0,1]$ in two different ways:

1) $t_i \in [x_i, x_{i+1}] \cap \mathbb{Q}$;

2) $t_i \in [x_i, x_{i+1}]$, and $t_i \notin \mathbb{Q}$.

Then the lower (with respect to the first way) and upper (with respect to the second way) Darboux sums are

$L(f,P) = \sum_{i=0}^{n-1} \inf_{t \in [x_i, x_{i+1}]} f(t)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) = 1 - 0 = 1;$

$U(f,P) = \sum_{i=0}^{n-1} \sup_{t \in [x_i, x_{i+1}]} f(t)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} 2(x_{i+1} - x_i) = 2 \cdot (1 - 0) = 2.$

We see that $\lim_{\max_{i \in [0, n-1]} (x_{i+1} - x_i) \to 0} \big(U(f,P) - L(f,P)\big) = 2 - 1 = 1 \neq 0$, so the function f is not Riemann integrable

(see https://www.encyclopediaofmath.org/index.php/Darboux_sum).

Remark. A slightly different way to prove this fact we can find here:

https://en.wikipedia.org/wiki/Riemann_integral#Examples .