Answer on Question #68446 – Math – Real Analysis

Question

If f, g is Riemann integrable on $[a, b]$ then fg is Riemann integrable on $[a, b]$.

Solution

Function is Riemann integrable on $[a, b]$ if and only if set $S_f = \{x \in [a, b] | f \text{ is not continuous at } x\}$ of points of discontinuity of f on $[a, b]$ has measure zero: $\mu(S_f) = 0$. Similarly, $\mu(S_g) = 0$.

If functions f and g are continuous at point x, then their product fg is continuous at x as well. Then fg may be not continuous at x when f or g is not continuous at x (but not always, for example multiplying not continuous function by total zero gives total zero, which is continuous). Thus, the set of discontinuities of fg is a subset of the sum of S_f and S_g: $S_{fg} \subset S_f \cup S_g$. Measure is additive function, so $\mu(S_{fg}) = \mu(S_f \cup S_g) \leq \mu(S_f) + \mu(S_g) = 0 + 0 = 0$. Thus, function fg is Riemann integrable on $[a, b]$.

Answer:

Yes, fg is Riemann integrable on $[a, b]$.