Answer on Question #68375 – Math – Differential Equations

Question

Show that u(x, y)=xf(2x+y) is a general solution of x∂u/∂x-2x ∂u/∂y=u

Solution

Substitute

$$u(x,y) = xf(2x+y)$$

into the original equation $x\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial y} = u$.

First we find $\frac{\partial u}{\partial x}$. Using the product rule we get

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\big(x \cdot f(2x+y)\big) = \frac{\partial}{\partial x}(x) \cdot f(2x+y) + x\frac{\partial}{\partial x}\big(f(2x+y)\big)$$

Now we use the chain rule for a derivative of the composite function

$$\frac{\partial u}{\partial x} = f(2x+y) + x\frac{\partial f(v)}{\partial v}\frac{\partial}{\partial x}(2x+y) = f(2x+y) + 2x\frac{\partial f(v)}{\partial v}$$

where $v = 2x + y$

Then we find $\frac{\partial u}{\partial y}$ using the chain rule

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\big(x \cdot f(2x+y)\big) = x\frac{\partial}{\partial y}\big(f(2x+y)\big) = x\frac{\partial f(v)}{\partial v}\frac{\partial}{\partial y}(2x+y) = x\frac{\partial f(v)}{\partial v}$$

Substituting $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ into the original equation

$$x\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial y} = u$$

gives

$$x\left(f(2x+y) + 2x\frac{\partial f(v)}{\partial v}\right) - 2x\left(x\frac{\partial f(v)}{\partial v}\right) = u(x,y)$$

or

$$xf(2x+y) + 2x^2\frac{\partial f(v)}{\partial v} - 2x^2\frac{\partial f(v)}{\partial v} = u \quad \Rightarrow \quad xf(2x+y) = u(x,y)$$

Thus, the function $u(x,y) = xf(2x+y)$ is a general solution of the equation

$$x\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial y} = u$$

Answer: $u(x,y) = xf(2x+y)$) is a general solution of $x\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial y} = u$.