Answer on Question #68374 – Math – Differential Equations

Question

Form a partial differential equation of

$$\varphi(x+y+u, x^2+y^2+u^2)=0$$

Solution

The solution of partial differential equation satisfies

$$\varphi(x+y+u, x^2+y^2+u^2)=0$$

for some arbitrary function φ and independent first integrals

$$x+y+u=c_1,$$
$$x^2+y^2+u^2=c_2.$$

Differentiating first integrals with respect to x

$$\begin{cases} 1+\frac{dy}{dx}+\frac{du}{dx}=0, \\ 2x+2y\frac{dy}{dx}+2u\frac{du}{dx}=0, \end{cases}$$

$$\begin{cases} \frac{dy}{dx}+\frac{du}{dx}=-1, \\ y\frac{dy}{dx}+u\frac{du}{dx}=-x. \end{cases}$$

Using Cramer’s rule

$$\frac{dy}{dx}=\frac{\begin{vmatrix} -1 & 1 \\ -x & u \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ y & u \end{vmatrix}}=\frac{-u+x}{u-y},$$

$$\frac{du}{dx}=\frac{\begin{vmatrix} 1 & -1 \\ y & -x \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ y & u \end{vmatrix}}=\frac{-x+y}{u-y}.$$

Then

$$\frac{dy}{\frac{-u+x}{u-y}}=dx,$$
$$\frac{du}{\frac{-x+y}{u-y}}=dx,$$

hence

$$dx = \frac{dy}{\frac{-u+x}{u-y}} = \frac{du}{\frac{-x+y}{u-y}}$$

that corresponds to the partial differential equation

$$\frac{\partial u}{\partial x} + \frac{-u+x}{u-y}\frac{\partial u}{\partial y} = \frac{-x+y}{u-y},$$

$$(u-y)\frac{\partial u}{\partial x} + (x-u)\frac{\partial u}{\partial y} = (y-x).$$

Answer: $(u-y)\frac{\partial u}{\partial x} + (x-u)\frac{\partial u}{\partial y} = (y-x).$