Answer on Question #68370 – Math – Differential Equations

Question

Form Partial differential eq. of x^2/a^2 +y^2/b^2 +u^2/c^2 =1

Solution

We have the following equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{u^2}{c^2}=1 \qquad (1)$$

Note that the number of constants (a, b, c) is more than the number of independent variables (x, y). Hence the order of the resulting differential equation will be more than 1.

Differentiating (1) partially with respect to x we get

$$\frac{2x}{a^2}+\frac{2u}{c^2}\frac{\partial u}{\partial x}=0 \qquad (2)$$

or

$$\frac{c^2}{a^2}=-\frac{u}{x}\frac{\partial u}{\partial x} \qquad (3)$$

Differentiating (2) partially with respect to x we get

$$\frac{1}{a^2}+\frac{1}{c^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+\frac{u}{c^2}\frac{\partial^2 u}{\partial x^2}=0$$

or

$$\frac{c^2}{a^2}=-\left(\frac{\partial u}{\partial x}\right)^2-u\frac{\partial^2 u}{\partial x^2} \qquad (4)$$

From (3) and (4) we get

$$xu\frac{\partial^2 u}{\partial x^2}+x\left(\frac{\partial u}{\partial x}\right)^2=u\frac{\partial u}{\partial x} \qquad (5)$$

which is required differential equation.

This equation is not unique. Similarly differentiating (1) partially with respect to y we get

$$\frac{2y}{b^2}+\frac{2u}{c^2}\frac{\partial u}{\partial y}=0 \qquad (6)$$

or

$$\frac{c^2}{b^2}=-\frac{u}{y}\frac{\partial u}{\partial y} \qquad (7)$$

and differentiating (6) partially with respect to y we get

$$\frac{1}{b^2}+\frac{1}{c^2}\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}+\frac{u}{c^2}\frac{\partial^2 u}{\partial y^2}=0$$

or

$$\frac{c^2}{b^2}=-\left(\frac{\partial u}{\partial y}\right)^2-u\frac{\partial^2 u}{\partial y^2} \qquad (8)$$

From (7) and (8) we get

$$yu\frac{\partial^2 u}{\partial y^2}+y\left(\frac{\partial u}{\partial y}\right)^2=u\frac{\partial u}{\partial y} \qquad (9)$$

which is also a required differential equation. Equations (5) and (9) can be summed so that the resulting equation is symmetric with respect to x and y

$$xu\frac{\partial^2 u}{\partial x^2}+yu\frac{\partial^2 u}{\partial y^2}+x\left(\frac{\partial u}{\partial x}\right)^2+y\left(\frac{\partial u}{\partial y}\right)^2=u\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}$$

Answer: The required partial differential equations are

$$xu\frac{\partial^2 u}{\partial x^2}+x\left(\frac{\partial u}{\partial x}\right)^2=u\frac{\partial u}{\partial x}$$

or

$$yu\frac{\partial^2 u}{\partial y^2}+y\left(\frac{\partial u}{\partial y}\right)^2=u\frac{\partial u}{\partial y}$$

or

$$xu\frac{\partial^2 u}{\partial x^2}+yu\frac{\partial^2 u}{\partial y^2}+x\left(\frac{\partial u}{\partial x}\right)^2+y\left(\frac{\partial u}{\partial y}\right)^2=u\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}$$