Answer on Question #68345 – Math – Linear Algebra

Question

use the Gaussian Elimination method to find the value of a so that the system of equations

$$\begin{cases} x+(a+4)y+(4a+2)z=0 \\ 2x+3ay+(3a+4)z=0 \\ x+2(a+1)y+(3a+4)z=0 \end{cases}$$

x+(a+4)y+(4a+2) z=0
2x+3ay+(3a+4) z=0
x+2(a+1)y+(3a+4) z=0
has **(i)** a unique solution ;
(ii) infinitely many solutions.
Further, find the solution set in each case.

Solution

$$\begin{cases} x+(a+4)y+(4a+2)z=0 \quad (R1) \\ 2x+3ay+(3a+4)z=0 \quad (R2) \\ x+2(a+1)y+(3a+4)z=0 \quad (R3) \end{cases}$$

$$\begin{cases} x+(a+4)y+(4a+2)z=0 & ((R1)' \leftarrow R1) \\ (a-8)y-5az=0 & ((R2)' \leftarrow R2-2R1) \\ (a-2)y+(-a+2)z=0 & ((R3)' \leftarrow R3-R1) \end{cases}$$

$$\begin{cases} x+(a+4)y+(4a+2)z=0 & ((R1)'' \leftarrow (R1)') \\ (a-8)y-5az=0 & ((R2)'' \leftarrow (R2)') \\ \dfrac{4a^2-16}{a-8}z=0 & ((R3)'' \leftarrow (R3)'-(R2)'\cdot\dfrac{a-2}{a-8}) \end{cases}$$

The matrix A for this linear system will be as follows:

$$A=\begin{pmatrix} 1 & a+4 & 4a+2 \\ 0 & a-8 & -5a \\ 0 & 0 & \dfrac{4a^2-16}{a-8} \end{pmatrix}$$

The determinant of the matrix A is given by

$$\det(A)=\begin{vmatrix} 1 & a+4 & 4a+2 \\ 0 & a-8 & -5a \\ 0 & 0 & \frac{4a^2-16}{a-8} \end{vmatrix}=1\cdot(a-8)\cdot\frac{4a^2-16}{a-8}=4a^2-16=4(a-2)(a+2).$$

Next,

$$\begin{cases} x+\left(4a+2+\dfrac{5a(a+4)}{a-8}\right)z=0 & ((R1)''' \leftarrow (R1)''-\dfrac{a+4}{a-8}\cdot(R2)'') \\ y-\dfrac{5a}{a-8}z=0 & ((R2)''' \leftarrow (R2)''\cdot\dfrac{1}{a-8}) \\ \dfrac{4a^2-16}{a-8}z=0 & ((R3)''' \leftarrow (R3)'') \end{cases}$$

The criterion says that the system has the unique trivial solution only if $\det(A)$ is not equal zero and infinitely many solutions otherwise.
So:
I. Unique solution $x = y = z = 0$ if $a \neq 2$ and $a \neq -2$;
II. Infinitely many solutions z=C , where C is an arbitrary real constant,
$y = \frac{5az}{a-8}$,
$x = -\left(4a + 2 + \frac{5a(a+4)}{a-8}\right)z = -\frac{9a^2-10a-16}{a-8}z$ if $a = 2$ or $a = -2$.
If $a = 2$, then $z = C$, where C is arbitrary real constant, $y = -\frac{5z}{3} = -\frac{5C}{3}$, $x = 0$.
If $a = -2$, then $z = D$, where D is arbitrary real constant, $y = -\frac{10z}{-10} = z = D$,
$x = \frac{-40z}{-10} = 4z = 4D$.