Answer on Question #67466 – Math – Complex Analysis

Question

Find the value of $a \in \mathbb{R}$ for which ai is a solution of

$$z^4 - 2z^3 + 7z^2 - 4z + 10 = 0.$$

Also find all the roots of this equation.

Solution

Let us substitute ai into the equation $z^4 - 2z^3 + 7z^2 - 4z + 10 = 0$ given that

$i^2 = -1;\ i^3 = -i;\ i^4 = 1$ (see https://en.wikipedia.org/wiki/Complex_number).

We get

$$a^4 + 2a^3 i - 7a^2 - 4ai + 10 = 0 \Leftrightarrow (a^4 - 7a^2 + 10) + (2a^3 - 4a)i = 0.$$

From the definition of equality of two complex numbers (see http://www.math-only-math.com/equality-of-complex-numbers.html) we conclude that

$$\begin{cases} a^4 - 7a^2 + 10 = 0 \\ 2a^3 - 4a = 0 \end{cases}.$$

Let us solve the second equation of the system:

$$2a(a^2 - 2) = 0 \Leftrightarrow \left[\begin{array}{l} a = 0 \\ a = \sqrt{2} \\ a = -\sqrt{2} \end{array}\right. .$$

But $a = 0$ does not satisfy the first equation.

Let us check $a = \sqrt{2}$: $4 - 7 \cdot 2 + 10 = 0 \Rightarrow a = \sqrt{2}$ is a solution of the obtained system.

Let us check $a = -\sqrt{2}$: $4 - 7 \cdot 2 + 10 = 0 \Rightarrow a = -\sqrt{2}$ is a solution of the obtained system.

So we have two roots of the original equation: $\left[\begin{array}{l} z_1 = i\sqrt{2} \\ z_2 = -i\sqrt{2} \end{array}\right.$ (corresponding to the values $\left[\begin{array}{l} a_1 = \sqrt{2} \\ a_2 = -\sqrt{2} \end{array}\right.$).

From the polynomial remainder theorem

(see https://en.wikipedia.org/wiki/Polynomial_remainder_theorem) it follows that the polynomial $z^4 - 2z^3 + 7z^2 - 4z + 10$ is divisible by polynomial $(z - i\sqrt{2})(z + i\sqrt{2}) = z^2 + 2$. Using long polynomial division (see https://en.wikipedia.org/wiki/Polynomial_long_division) we obtain

$\frac{z^4-2z^3+7z^2-4z+10}{z^2+2} = z^2 - 2z + 5.$

To solve the equation $z^2 - 2z + 5 = 0$ we apply the quadratic formula (see https://en.wikipedia.org/wiki/Quadratic_formula) and obtain that

$$\left[\begin{array}{l} z_3 = 1 + 2i \\ z_4 = 1 - 2i \end{array}\right.$$

So all the roots of the original equation are

$$\left[\begin{array}{c} z_1 = i\sqrt{2} \\ z_2 = -i\sqrt{2} \\ z_3 = 1 + 2i \\ z_4 = 1 - 2i \end{array}\right..$$

Answer: $a = \pm\sqrt{2}; z = \pm i\sqrt{2}, z = 1 \pm 2i.$