Answer on Question #67465 – Math – Linear Algebra

Question

Let V be the set of all functions that are twice differentiable in $\mathbb{R}$ and

$S = \{\cos x, \sin x, x\cos x, x\sin x\}$.

a) Check that S is a linearly independent set over $\mathbb{R}$. (Hint: Consider the equation

$$a_0\cos x + a_1\sin x + a_2 x\cos x + a_3 x\sin x.$$

Put $x = 0, \pi, \frac{\pi}{2}, \frac{\pi}{4}$ etc and solve for a_i.)

b) Let $V = [S]$ and let $T: V \to V$ be the function defined by $T\big(f(x)\big) = \frac{\partial^2}{\partial x^2} f(x) + 2\frac{\partial}{\partial x} f(x)$. Check that T is a linear transformation on V.

Solution

a) $a_0\cos x + a_1\sin x + a_2 x\cos x + a_3 x\sin x = 0$.

For $x = 0$:

$$\begin{aligned}
&a_0\cos 0 + a_1\sin 0 + a_2\cdot 0\cdot\cos 0 + a_3\cdot 0\cdot\sin 0 = 0 \to\\
&a_0 + 0a_1 + 0a_2 + 0a_3 = 0 \to\\
&a_0 = 0.
\end{aligned}$$

For $x = \pi$:

$$\begin{aligned}
&a_1\sin\pi + a_2\cdot\pi\cdot\cos\pi + a_3\cdot\pi\cdot\sin\pi = 0 \to\\
&0a_1 - \pi a_2 + 0a_3 = 0 \to\\
&a_2 = 0.
\end{aligned}$$

For $x = \frac{\pi}{2}$:

$$\begin{aligned}
&a_1\sin\frac{\pi}{2} + a_3\cdot\frac{\pi}{2}\cdot\sin\frac{\pi}{2} = 0 \to\\
&a_1 + \frac{\pi}{2}a_3 = 0 \to\\
&a_1 = -\frac{\pi}{2}a_3.
\end{aligned}$$

For $x = \frac{\pi}{4}$:

$$\begin{aligned}
&\left(-\frac{\pi}{2}a_3\right)\sin\frac{\pi}{4} + a_3\cdot\frac{\pi}{4}\cdot\sin\frac{\pi}{4} = 0 \to\\
&\frac{2}{\sqrt{2}}\left(-\frac{\pi}{2}a_3\right) + \frac{\pi}{4}\cdot\frac{2}{\sqrt{2}}a_3 = 0 \to\\
&-\frac{\pi}{2}a_3 + \frac{\pi}{4}a_3 = 0 \to\\
&-\frac{\pi}{4}a_3 = 0 \to\\
&a_3 = 0, \qquad a_1 = -\frac{\pi}{2}a_3 = 0.
\end{aligned}$$

Thus, over $\mathbb{R}$ the equation $a_0\cos x + a_1\sin x + a_2 x\cos x + a_3 x\sin x = 0$ has only the trivial solution $a_0 = a_1 = a_2 = a_3 = 0$, therefore S is a linearly independent set over $\mathbb{R}$.

b) Linear transformation is such that

$$T(kf) = kT(f), k \in \mathbb{R}$$

and

$$T(f + g) = T(f) + T(g).$$

Since differentiation is a linear operation

$$(u + v)' = u' + v' \text{ and } (ku)' = ku',$$

transformation T is also linear.