Answer on Question # 67333 - Math - Differential Equations

Question

The differential equation y'''-y''=8x^2. I know after solving the homogeneous part of the equation you get that 1 and 0 are roots of the equation. I know for the 8x^2 you're supposed to substitute the constant for a variable then that would be Ax^2 and that i have to derive three times, My question is, after i derive three times what would i do with the second and the first derivative? there isn't any y'' or y' where to replace that in the original equation.

Solution

To solve a nonhomogeneous linear differential equation (DE)

$$y''' - y'' = 8x^2 \tag{1}$$

we must:

1) find the complementary function y_c that is the general solution of the associated homogeneous DE

$$y''' - y'' = 0$$

2) find any particular solution y_p of the nonhomogeneous equation

$$y''' - y'' = 8x^2$$

The general solution of the equation (1) is

$$y = y_c + y_p$$

To solve the associated homogeneous DE

$$y''' - y'' = 0 \tag{2}$$

we substitute into equation the solution as exponential function $y = e^{mx}$. We get

$$m^3 e^{mx} - m^2 e^{mx} = 0$$

or

$$(m^3 - m^2)e^{mx} = 0$$

This equation is satisfied only when *m* is a solution or a root of the third-degree polynomial equation

$$m^3 - m^2 = 0$$

$$m^2(m-1) = 0$$

Its roots are $m_1 = m_2 = 0,\ \ m_3 = 1,$ and the general solution of the associated homogeneous DE is

$$y_c = C_1 + C_2 x + C_3 e^x$$

where $C_1,\ C_2,\ C_3$ are real constants.

Now go to your question.

First, $y_p = Ax^2$ is not a correct form of a particular solution, you could regard

$$y_p = Ax^2 + Bx + C$$

where $A,\ B, C$ are the undetermined coefficients.

However, for this problem it is also incorrect. The correct solution has the form

$$y_p = x^2(Ax^2 + Bx + C) = Ax^4 + Bx^3 + Cx^2$$

This is due to the fact that the right-hand side of equation is $8x^2 = 8x^2 e^0$, i.e., the exponent is zero. Since the root of the characteristic equation is also zero, $m_1 = m_2 = 0$, and this is the double root, then the factor x^2 is obligatory.

Now solve the equation. Since the equation include y''_p and y'''_p we first find derivatives

$$y'_p = 4Ax^3 + 3Bx^2 + 2Cx$$

$$y''_p = 12Ax^2 + 6Bx + 2C$$

$$y'''_p = 24Ax + 6B$$

Substitute $y'''_p = 24Ax + 6B$ and $y''_p = 12Ax^2 + 6Bx + 2C$ into the equation (1). Since the equation (1) does not include y', we simply do not pay attention to it. We get

$$24Ax + 6B - (12Ax^2 + 6Bx + 2C) = 8x^2$$

or

$$24Ax + 6B - 12Ax^2 - 6Bx - 2C = 8x^2$$

We now define $A,\ B, C$ by solving the system of equations

$$\begin{cases} -12Ax^2 = 8x^2 \\ 24Ax - 6Bx = 0 \\ 6B - 2C = 0 \end{cases}$$

or

$$\begin{cases} 12A = -8 \\ 24A = 6B \\ 6B = 2C \end{cases}$$

The solution is $A = -\frac{2}{3},\ \ B = -\frac{8}{3},\ \ C = -8$ and

$$y_p = -\frac{2}{3}x^4 - \frac{8}{3}x^3 - 8x^2$$

Finally we get the general solution of the equation

$$y = C_1 + C_2x + C_3e^x - \frac{2}{3}x^4 - \frac{8}{3}x^3 - 8x^2$$

Answer: the general solution of the equation is

$$y = C_1 + C_2x + C_3e^x - \frac{2}{3}x^4 - \frac{8}{3}x^3 - 8x^2$$