Answer on Question #67205 – Math – Differential Equations

Question

Show that the function
i) u(x, t) = A(x + ct)^3 is a solution of the one-dimensional wave equation
ii) u(x, t) =exp(-ut) sin x is a solution of the one-dimensional heat equation

Solution

i)

One-dimensional wave equation is

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

To show that the function

$$u(x,t) = A(x+ct)^3$$

is a solution of the one-dimensional wave equation we first find $\frac{\partial^2 u(x,t)}{\partial t^2}$ and $\frac{\partial^2 u(x,t)}{\partial x^2}$ for the given function:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial t}[A(x+ct)^3] = 3Ac(x+ct)^2$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial t}[3Ac(x+ct)^2] = 6Ac^2(x+ct)$$

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial}{\partial x}[A(x+ct)^3] = 3A(x+ct)^2$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial}{\partial x}[3A(x+ct)^2] = 6A(x+ct)$$

Substituting the obtained values into the one-dimensional wave equation one gets the identity

$$6Ac^2(x+ct) = c^2 6A(x+ct)$$

Thus, the function $u(x,t) = A(x+ct)^3$ is a solution of the one-dimensional wave equation.

ii)

One-dimensional heat equation is

$$\frac{\partial u(x,t)}{\partial t} = u\frac{\partial^2 u(x,t)}{\partial x^2},$$

where $u(x,t)$ is the function, u is the constant.

To show that given function is a solution of the one-dimensional heat equation we first find $\frac{\partial u(x,t)}{\partial t}$ and $\frac{\partial^2 u(x,t)}{\partial x^2}$ for the given function

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial t}[\exp(-ut)\sin x] = -u\exp(-ut)\sin x$$

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial}{\partial x}[\exp(-ut)\sin x] = \exp(-ut)\cos x$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial}{\partial x}[\exp(-ut)\cos x] = -\exp(-ut)\sin x$$

Substituting the obtained values into the one-dimensional heat equation one gets the identity

$$-u\exp(-ut)\sin x = -u\exp(-ut)\sin x$$

Thus, the function $u(x,t) = \exp(-ut)\sin x$ is a solution of the one-dimensional heat equation.

Answer:

i) u(x, t) = A(x + ct)^3 is a solution of the one-dimensional wave equation;
ii) u(x, t)=exp(-ut)sinx is a solution of the one-dimensional heat equation.