Answer on Question #67036 – Math – Algebra

Question

Prove De Moivre's theorem for $n \in \mathbb{Z}\backslash\mathbb{N}$.

Solution

So we need to prove that

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx \text{ for } n \in \mathbb{Z}\backslash\mathbb{N}.$$

The set $\mathbb{Z}\backslash\mathbb{N}$ consists of all negative integers and zero. Our proof will be based on proof for natural n using the method of math induction.

It is clear that formula holds for $n = 1$. Assuming it holds for some natural $n = k$, that is,

$$(\cos x + i \sin x)^k = \cos kx + i \sin kx,$$

we'll prove that it also holds for $n = k + 1$. By induction hypothesis and using that $i^2 = -1$, we have

$$(\cos x + i \sin x)^{k+1} = (\cos x + i \sin x)^k (\cos x + i \sin x) = (\cos kx + i \sin kx)(\cos x + i \sin x)$$
$$= \cos kx \cos x - \sin kx \sin x + i(\cos kx \sin x + \sin kx \cos x).$$

Then using angle sum and difference identities

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

we simplify the previous expression, and obtain

$$(\cos x + i \sin x)^{k+1} = \cos(k+1)x + i \sin(k+1)x.$$

Hence, by the principle of math induction, De Moivre's identity holds for all $n \in \mathbb{N}$.

Now we proceed to the second part of the proof, let's show that identity holds for all negative integers and zero. The case $n = 0$ is satisfied, since $\cos(0) + i \sin(0) = 1 = (\cos x + i \sin x)^0$.

In case of negative integers we'll use the statement for natural n. If $n \in \mathbb{N}$, then $-n$ will be a negative integer, so

$$(\cos x + i \sin x)^{-n} = \frac{1}{(\cos x + i \sin x)^n} = \frac{1}{\cos nx + i \sin nx}$$

Finally, we multiple and divide the right-hand side of fraction by $(\cos nx - i \sin nx)$:

$$(\cos x + i\sin x)^{-n} = \frac{\cos nx - i\sin nx}{(\cos nx + i\sin nx)(\cos nx - i\sin nx)} = \frac{\cos nx - i\sin nx}{\cos^2 nx + \sin^2 nx}$$
$$= \frac{\cos nx - i\sin nx}{1} = \cos nx - i\sin nx,$$

thus proving the identity for negative integers.

So we proved that identity holds for $n \in \mathbb{Z}\backslash\mathbb{N}$ (using that it also holds for $n \in \mathbb{N}$).