Answer on Question #66999 – Math – Discrete Mathematics

Question

Let $a_n \geq \frac{2^n - 1}{n}$ and let $\nu \leq [\log_2 n]$. Show that $a_n - \nu > 0$.

Solution

Since $a_n \geq \frac{2^n - 1}{n}$ and $\nu \leq [\log_2 n]$, it is sufficient to establish that

$$\frac{2^n - 1}{n} > [\log_2 n]$$

for each n, then

$$a_n - \nu > 0.$$

For $n = 1$ we have that $\frac{2^1 - 1}{1} > [\log_2 1] \Leftrightarrow 1 > 0$ holds true.

For $n = 2$ we have that $\frac{2^2 - 1}{2} > [\log_2 2] \Leftrightarrow \frac{3}{2} > 1$ holds true.

For $n = 3$ we have that $\frac{2^3 - 1}{3} > [\log_2 3] \Leftrightarrow \frac{7}{3} > 1$ holds true.

For $n = 4$ we have that $\frac{2^4 - 1}{4} > [\log_2 4] \Leftrightarrow \frac{15}{4} > 2$ holds true.

Let $n \geq 5$. Then by binomial theorem and $\frac{n-2}{3} \geq 1$ we have that

$$\frac{2^n - 1}{n} = \frac{(1+1)^n - 1}{n} = \frac{\left(1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} + \ldots + \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)}{2} + n + 1\right) - 1}{n} \geq$$

$$\geq \frac{\left(1 + n + 2\frac{n(n-1)(n-2)}{6}\right) - 1}{n} = \frac{n + n(n-1)\frac{(n-2)}{3}}{n} \geq \frac{n + n(n-1)}{n} = \frac{n + n^2 - n}{n} =$$

$$= n \geq \log_2 n \geq [\log_2 n].$$

Hence $\frac{2^n - 1}{n} > [\log_2 n]$ for each n.