Answer on Question #66843 - Math – Differential Equations

Question

Obtain the Fourier series for the following periodic function which has a period of 2π: f(x) = x^2 for -π ≤x ≤π.

Solution

For the periodic function $f(x) = x^2$ which has a period of 2π, $-\pi \le x \le \pi$, the Fourier series is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

are the Fourier coefficients.

Find a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{1}{3\pi}(\pi^3 - (-\pi)^3) = \frac{2\pi^2}{3}$$

Find a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(nx)dx$$

here we used that the integrand, $x^2\cos(nx)$, is even. To find a_n we use integration by parts

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx = \frac{2}{\pi n}\int_0^{\pi} x^2 d(\sin(nx)) = \frac{2}{\pi n}x^2\sin(nx)|_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi} 2x\sin(nx)dx =$$

$$= \frac{2}{\pi n}(\pi^2\sin(n\pi) - 0) - \frac{4}{\pi n^2}\int_0^{\pi} xd(-\cos(nx)) = 0 + \frac{4}{\pi n^2}\int_0^{\pi} xd(\cos(nx)) =$$

$$= \frac{4}{\pi n^2}x\cos(nx)|_0^{\pi} - \frac{4}{\pi n^2}\int_0^{\pi}\cos(nx)dx = \frac{4}{\pi n^2}(\pi\cos(n\pi) - 0) - \frac{4}{\pi n^3}\sin(nx)|_0^{\pi} =$$

$$= \frac{4}{n^2}\cos(n\pi) - \frac{4}{\pi n^3}(\sin(n\pi) - \sin 0) = \frac{4}{n^2}(-1)^n - 0 = (-1)^n\frac{4}{n^2}$$

Find b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)dx = 0$$

since the integrand, $x^2\sin(nx)$, is odd.

So

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Answer: Fourier series is

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$