ANSWER on Question #66814, Math / Abstract Algebra

Consider the set $X = R\{-1\}$. Define $*$ on X by

$$x_1 * x_2 = x_1 + x_2 + x_1 \times x_2, \quad \forall\, x_1\,, x_2\; \in X$$

1) check whether $(X,*)$ is a group or not.

2) Prove that $\underbrace{x * x * x * \cdots * x}_{n\ times} = (1+x)^n - 1, \quad \forall n \in \mathbb{N}\ \ and\ \forall x \in X$

SOLUTION

1) By the definition, the $(X,*)$ is a group If three conditions are satisfied

a) $\forall(a, b, c \in X)$: $(a * b) * c = a * (b * c)$

b) $\exists e \in X\ \ \forall a \in X\!: a * e = e * a = a - identity\ element$

c) $\forall a \in X\ \ \exists a^{-1} \in X :\ a * a^{-1} = a^{-1} * a = e - inverse\ element$

We begin to check the conditions. We will do this in order

a)

$$(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$$

$$(x_1 * x_2) * x_3 = (x_1 + x_2 + x_1 \times x_2) * x_3 =$$

$$= x_1 + x_2 + x_1 \times x_2 + x_3 + (x_1 + x_2 + x_1 \times x_2) \times x_3 =$$

$$= x_1 + x_2 + x_1 \times x_2 + x_3 + x_1 \times x_3 + x_2 \times x_3 + x_1 \times x_2 \times x_3 =$$

$$= x_1 + x_2 + x_3 + x_1 \times x_2 + x_1 \times x_3 + x_2 \times x_3 + x_1 \times x_2 \times x_3$$

On the other hand

$$x_1 * (x_2 * x_3) = x_1 * (x_2 + x_3 + x_2 \times x_3) =$$

$$= x_1 + x_2 + x_3 + x_2 \times x_3 + x_1 \times (x_2 + x_3 + x_2 \times x_3) =$$

$$= x_1 + x_2 + x_3 + x_2 \times x_3 + x_1 \times x_2 + x_1 \times x_3 + x_1 \times x_2 \times x_3$$

As we can see

$$(x_1 * x_2) * x_3 = x_1 + x_2 + x_3 + x_1 \times x_2 + x_1 \times x_3 + x_2 \times x_3 + x_1 \times x_2 \times x_3 =$$

$$= x_1 * (x_2 * x_3)$$

Conclusion, the first condition is satisfied

b) $\exists e \in X \ \ \forall a \in X: a * e = e * a = a - identity\ element$

As an identity element, we choose $e = 0$

We check that the condition is satisfied with such a choice of an identity element

$$a * 0 = a + 0 + a \times 0 = a + 0 + 0 = a$$

$$0 * a = 0 + a + 0 \times a = 0 + a + 0 = a$$

As we can see,

$$a * 0 = 0 * a = a$$

What corresponds to the definition of an identity element.

Conclusion, the second condition is satisfied

c) $\forall a \in X \ \ \exists a^{-1} \in X: \ a * a^{-1} = a^{-1} * a = e - inverse\ element$

Our task is to find the inverse element. For this we use the definition

$$a * a^{-1} = e$$

As was shown in b), the neutral element in this "group" is zero

Then,

$$a * a^{-1} = a + a^{-1} + a \times a^{-1} = 0 \leftrightarrow a^{-1}(1 + a) = -a \leftrightarrow a^{-1} = -\frac{a}{a+1}$$

$$a^{-1} = -\frac{a}{a+1} \in X$$

It remains to verify the execution of its properties

$$a * a^{-1} = a * \left(-\frac{a}{a+1}\right) = a + \left(-\frac{a}{a+1}\right) + a \times \left(-\frac{a}{a+1}\right) =$$

$$= a - \frac{a}{a+1} - \frac{a^2}{a+1} = \frac{a(a+1)}{a+1} - \frac{a}{a+1} - \frac{a^2}{a+1} = \frac{a^2 + a - a - a^2}{a+1} = 0$$

$$a^{-1} * a = \left(-\frac{a}{a+1}\right) * a = -\frac{a}{a+1} + a + \left(-\frac{a}{a+1}\right) \times a =$$

$$= -\frac{a}{a+1} + \frac{a(a+1)}{a+1} - \frac{a^2}{a+1} = \frac{-a + a^2 + a - a^2}{a+1} = 0$$

Conclusion, the third condition is satisfied

ANSWER

$(X,*)$ is a group

2) Prove that $\underbrace{x * x * x * \cdots * x}_{n\ times} = (1+x)^n - 1, \quad \forall n \in \mathbb{N}\ and\ \forall x \in X$

This formula will be proved by the method of mathematical induction

1 step: Basis of induction

$n = 1$

$$\underbrace{x * x * x * \cdots * x}_{n\ times} = (1+x)^n - 1 \leftrightarrow \begin{cases} (1+x)^1 - 1 = 1 + x - 1 = x \\ \underbrace{x * x * x * \cdots * x}_{n\ times} = x \end{cases}$$

As we can see

$$\underbrace{x * x * x * \cdots * x}_{1\ times} = x = (1+x)^1 - 1$$

Conclusion, the formula is true for $n = 1$

$n = 2$

$$\underbrace{x * x * x * \cdots * x}_{n\ times} = (1+x)^n - 1 \leftrightarrow \begin{cases} (1+x)^2 - 1 \\ \underbrace{x * x * x * \cdots * x}_{n\ times} = x * x \end{cases}$$

$$(1+x)^2 - 1 = 1 + 2x + x^2 - 1 = 2x + x^2$$

$$\underbrace{x * x}_{by\ the\ definition} = x + x + x \times x = 2x + x^2$$

As we can see

$$\underbrace{x * x * x * \cdots * x}_{2\ times} = 2x + x^2 = (1+x)^2 - 1$$

Conclusion, the formula is true for $n = 2$

2 step: Induction hypothesis

Suppose that the formula is true for $n = k,\ \forall k \in \mathbb{N}$

$$\underbrace{x * x * x * \cdots * x}_{k\ times} = (1 + x)^k - 1$$

3 step: Inductive transition

It is necessary to prove that the formula is true for $n = k + 1,\ \forall k \in \mathbb{N}$,

$$\underbrace{x * x * x * \cdots * x}_{k+1\ times} = (1 + x)^{k+1} - 1$$

using the inductive hypothesis

In our case,

$$\underbrace{x * x * x * \cdots * x}_{k+1\ times} = \left(\underbrace{x * x * x * \cdots * x}_{k\ times}\right) * x = ((1 + x)^k - 1) * x =$$

$$= \begin{pmatrix} x_1 = (1 + x)^k - 1 \\ x_2 = x \end{pmatrix} = x_1 + x_2 + x_1 \times x_2 =$$

$$= (1 + x)^k - 1 + x + ((1 + x)^k - 1) \times x = (1 + x)^k - 1 + x + x \times (1 + x)^k - x =$$

$$= 1 \times (1 + x)^k + x \times (1 + x)^k - 1 + x - x = (1 + x)^k \times (1 + x) - 1 =$$

$$= (1 + x)^{k+1} - 1$$

Conclusion,

$$\underbrace{x * x * x * \cdots * x}_{k+1\ times} = (1 + x)^{k+1} - 1$$

Q.E.D