Answer on Question #66803–Math–Linear Algebra

Question

Apply the Gram-Schmidt orthogonalization process to find an orthonormal basis for the subspace of R^4 generated by the vectors {(-1,1,0,1),(1,0 ,-1,0),(1,0,2,-1)}.

Solution

Let $u_1 = (-1,1,0,1)$, $u_2 = (1,0\ ,-1,0)$, $u_3 = (1,0,2,-1)$.

By the Gram-Schmidt orthogonalization process ([**1**], [**2**, pp 544, 558]) consider $v_1 = (-1,1,0,1)$ and $v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$;

Note that

$u_2 \cdot v_1 = 1 \cdot (-1) + 0 \cdot 1 + (-1) \cdot 0 + 0 \cdot 1 = -1$;

$v_1 \cdot v_1 = \|v_1\| = (-1) \cdot (-1) + 1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 = 3$.

$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (1,0\ ,-1,0) - \frac{(-1)}{3}(-1,1,0,1) = (1,0\ ,-1,0) + \frac{1}{3}(-1,1,0,1) = \frac{1}{3}(2,1,-3,1)$,

we can to consider $v_2 = (2,1,-3,1)$.

Next calculate $v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$;

$u_3 \cdot v_1 = 1 \cdot (-1) + 0 \cdot 1 + 2 \cdot 0 + (-1) \cdot 1 = -2$;

$u_3 \cdot v_2 = 1 \cdot 2 + 0 \cdot 1 + 2 \cdot (-3) + (-1) \cdot 1 = -5$;

$v_2 \cdot v_2 = 2 \cdot 2 + 1 \cdot 1 + (-3) \cdot (-3) + 1 \cdot 1 = 15$.

We get $v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2 = (1,0,2,-1) - \frac{(-2)}{3}(-1,1,0,1) - \frac{(-5)}{15}(2,1,-3,1) =$

$= (1,0,2,-1) + \frac{2}{3}(-1,1,0,1) + \frac{1}{3}(2,1,-3,1) = (1,1,1,0)$, hence, $v_3 = (1,1,1,0)$.

So we get an orthogonal basis $v_1 = (-1,1,0,1)$, $v_2 = (2,1,-3,1)$, $v_3 = (1,1,1,0)$.

Normalizing the vectors:

$\|v_1\| = \sqrt{v_1 \cdot v_1} = \sqrt{3};\ e_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}(-1,1,0,1).$

$\|v_2\| = \sqrt{v_2 \cdot v_2} = \sqrt{15};\ e_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{15}}(2,1,-3,1).$

$\|v_3\| = \sqrt{v_3 \cdot v_3} = \sqrt{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0} = \sqrt{3};\ e_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{3}}(1,1,1,0)$.

Answer: $\frac{1}{\sqrt{3}}(-1,1,0,1)$; $\frac{1}{\sqrt{15}}(2,1,-3,1)$ $\frac{1}{\sqrt{3}}(1,1,1,0)$.

References:

1 Hazewinkel, Michel (2001) Orthogonalization. Encyclopedia of Mathematics, Springer ISBN 978-1-55608-010-4 https://www.encyclopediaofmath.org/index.php/Orthogonalization

2 E W. Cheney,D. Ronald (2009) Linear Algebra: Theory and Applications, Sudbury Massachusetts, 740 p. ISBN 978-0-7637-5020-6.

https://books.google.com/books?id=Gg3Uj1GkHK8C&pg=PA544&redir_esc=y#v=onepage&q&f=false