Answer on Question #66802, Math, Linear Algebra

Let $T: R^4 \rightarrow R^4$ be defined by

$$T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$$

Check whether T is a linear operator and $T^4 = I$. Is T invertible?

Solution.

For linear operator A:

$$A(\alpha x + \alpha' x') = \alpha A(x) + a' A(x')$$

where x and x' are vectors in R^4.

We have:

$$x = (x_1, x_2, x_3, x_4) \;\; ; \quad x' = (-x_2, x_1, -x_4, x_3)$$

$$T(\alpha x + \alpha' x') = T(\alpha x_1 - \alpha' x_2, \alpha x_2 + \alpha' x_1, \alpha x_3 - \alpha' x_4, \alpha x_4 + \alpha' x_3) =$$

$$= (-\alpha x_2 - \alpha' x_1, \alpha x_1 - \alpha' x_2, -\alpha x_4 - \alpha' x_3, \alpha x_3 - \alpha' x_4)$$

$$\alpha T(x) = \alpha(-x_2, x_1, -x_4, x_3)$$

$$a' T(x') = a'(-x_1, -x_2, -x_3, -x_4)$$

$$(-\alpha x_2 - \alpha' x_1, \alpha x_1 - \alpha' x_2, -\alpha x_4 - \alpha' x_3, \alpha x_3 - \alpha' x_4) = \alpha(-x_2, x_1, -x_4, x_3) + a'(-x_1, -x_2, -x_3, -x_4)$$

So far, T is a linear operator.

$$T^2(x_1, x_2, x_3, x_4) = (-x_1, -x_2, -x_3, -x_4)$$

$$T^3(x_1, x_2, x_3, x_4) = (x_2, -x_1, x_4, -x_3)$$

$$T^4(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)$$

$$T^4(x_1, x_2, x_3, x_4) = I(x_1, x_2, x_3, x_4)$$

$$T^4 = I$$

We can write:

$$T^{-1}(-x_2, x_1, -x_4, x_3) = (x_1, x_2, x_3, x_4)$$

Then:

$$TT^{-1}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)$$

$$T^{-1}T(x_1, x_2, x_3, x_4) = T(x_2, -x_1, x_4, -x_3) = (x_1, x_2, x_3, x_4)$$

$$TT^{-1} = T^{-1}T = I$$

T is invertible.