Answer on Question #66785 – Math – Algorithms | Quantitative Methods

Question

Determine the spacing h in a table of equally spaced values for the function

$$f(x) = (x+2)^4 \ , \ \ 1 \le x \le 2$$

so that the quadratic interpolation in this table satisfies $|error| \le 10^{-6}$.

Solution

$$f(x) - P_n(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!} f^{(n+1)}(c_x)$$

where $P_n(x)$ is the polynomial of degree less than or equal to n interpolating the f at the $n+1$ data points $x_0, x_1, x_2, \ldots, x_n$ in $[a,b]$; c_x is between the maximum and minimum of $x, x_0, x_1, x_2, \ldots, x_n$.

We have $n = 2$. Then:

$$f(x) - P_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f'''(c_x)$$

$$f'(x) = 4(x+2)^3$$

$$f''(x) = 12(x+2)^2$$

$$f'''(x) = 24(x+2)$$

$$f(x) - P_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{6} \cdot 24(c_x+2)$$

where $1 \le x \le 2$, c_x is between the maximum and minimum of x, x_0, x_1, x_2.

If we assume that $x_0 \le x \le x_2$ and that $h = x_1 - x_0 = x_2 - x_1$, then c_x is between x_0 and x_2 , and we have:

$$|f(x) - P_2(x)| \le \left| \frac{(x-x_0)(x-x_1)(x-x_2)}{6} \cdot 24(x_2+2) \right| \le 96 \left| \frac{(x-x_0)(x-x_1)(x-x_2)}{6} \right|$$

If $h = x_1 - x_0 = x_2 - x_1$, then

$$\max_{x_0 \le x \le x_2} \left| \frac{(x-x_0)(x-x_1)(x-x_2)}{6} \right| = \frac{h^3}{9\sqrt{3}},$$

$$|error| = |f(x) - P_2(x)| \le \frac{h^3}{9\sqrt{3}} \cdot 96.$$

Thus,

$$\frac{h^3}{9\sqrt{3}} \cdot 96 = 10^{-6},$$

$$h = \frac{\sqrt[3]{3\sqrt{3}/32}}{100} \approx 0.0055.$$

Answer: $h = \frac{\sqrt[3]{3\sqrt{3}/32}}{100} \approx 0.0055.$