Answer on Question #66661 – Math – Differential Equations

Question

Find the integrating factor of the differential equation

(6xy-3y^2+2y) dx + 2(x-y) dy =0

and hence solve it.

Solution

$\mathrm{P}(\mathrm{x},\mathrm{y})\mathrm{dx}+\mathrm{Q}(\mathrm{x},\mathrm{y})\mathrm{dy}=0$, where

$\mathrm{P}(\mathrm{x},\mathrm{y})=6\mathrm{xy}-3\mathrm{y}^2+2\mathrm{y}$ and $\mathrm{Q}(\mathrm{x},\mathrm{y})=2(\mathrm{x}-\mathrm{y})$

First, we need to check whether

$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$.

$$\frac{\partial P}{\partial y}=6x-6y+2\neq 2=\frac{\partial Q}{\partial x};$$

$$\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=6x-6y+2-2=6x-6y=6(x-y)$$

It can be noticed that

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{6(x-y)}{2(x-y)}=3$$

that means our integrating factor as a function depends on x only:

$$\mu=\mu(x).$$

We can find it from the equation:

$$\frac{1}{\mu}\frac{d\mu}{dx}=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=3$$

$$\frac{d\mu}{\mu}=3dx$$

$$\int\frac{d\mu}{\mu}=3\int dx$$

$$\ln|\mu|=3x+\ln|C|$$

$$\mu=Ce^{3x},$$

but we will take the integrating factor

$$\mu=e^{3x}$$

Now, we have

$$e^{3x}(6xy-3y^2+2y)dx+2e^{3x}(x-y)dy=0$$
$$P(x,y)=e^{3x}(6xy-3y^2+2y)\text{and } Q(x,y)=2e^{3x}(x-y)$$
$$\frac{\partial P}{\partial y}=e^{3x}(6x-6y+2)=e^{3x}(-6y+2+6x)=\frac{\partial Q}{\partial x}$$
$$\text{So}, \exists u(x,y): \begin{cases} \frac{\partial u}{\partial x}=e^{3x}(6xy-3y^2+2y) \\ \frac{\partial u}{\partial y}=2e^{3x}(x-y) \end{cases}$$
$$u(x,y)=\int e^{3x}(6xy-3y^2+2y)dx+\varphi(y)$$
$$=\frac{1}{3}\int 6xyd(e^{3x})+\frac{1}{3}e^{3x}(-3y^2+2y)+\varphi(y)=$$
$$=e^{3x}2xy-\frac{1}{3}\int e^{3x}d(6xy)+\frac{1}{3}e^{3x}(-3y^2+2y)+\varphi(y)=$$
$$=e^{3x}2xy-2\int e^{3x}ydx+\frac{1}{3}e^{3x}(-3y^2+2y)+\varphi(y)=$$
$$=e^{3x}\left(2xy-\frac{2}{3}y-y^2+\frac{2}{3}y\right)+\varphi(y)=e^{3x}(2xy-y^2)+\varphi(y)$$
$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(e^{3x}(2xy-y^2)+\varphi(y)\right)=2e^{3x}(x-y)$$
$$e^{3x}(2x-2y)+\varphi'(y)=2e^{3x}(x-y)$$
$$\varphi'(y)=0$$
$$\varphi(y)=C$$
$$u(x,y)=e^{3x}(2xy-y^2)+C=0,$$
where C is an arbitrary real constant.

Answer: $u(x,y)=e^{3x}(2xy-y^2)+C=0.$