Answer on Question #66660 – Math – Differential Equations

Question

Solve the following equation by changing the independent variable

(1+x^2)^2 y" +2x(1+x^2) y' + 4y=0

Solution

We have equation

$$(1+x^2)^2 y'' + 2x(1+x^2)y' + 4y = 0$$

Divide both sides of the equation by $(1+x^2)$

$$(1+x^2)y'' + 2xy' + \frac{4y}{(1+x^2)} = 0$$

For the first and the second terms we have

$$(1+x^2)y'' + 2xy' = \left((1+x^2)y'\right)'$$

The equation takes the form

$$\left((1+x^2)y'\right)' + \frac{4y}{(1+x^2)} = 0$$

or

$$(1+x^2)\frac{d}{dx}\left((1+x^2)\frac{dy}{dx}\right) + 4y = 0$$

Change the independent variable x by $t = \arctan x$. Replace the variable in the derivative

$$\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx} = \frac{d}{dt}\frac{d(\arctan x)}{dx} = \frac{d}{dt} \cdot \frac{1}{1+x^2}$$

Multiplying both sides of equality by $1+x^2$ we get

$$(1+x^2)\frac{d}{dx} = \frac{d}{dt}$$

The equation takes the form

$$\frac{d^2y}{dt^2} + 4y = 0$$

The solution of this equation is

$$y = A\cos 2t + B\sin 2t$$

where A and B are arbitrary real constants.

Now we change the variable t by x using the formula

$$\cos 2t = \frac{1 - \tan^2 t}{1 + \tan^2 t}, \quad \sin 2t = \frac{2\tan t}{1 + \tan^2 t}$$

Since $x = \tan t$ we get

$$\cos 2t = \frac{1 - x^2}{1 + x^2}, \quad \sin 2t = \frac{2x}{1 + x^2}$$

Then the solution of the original equation is

$$y = A\frac{1 - x^2}{1 + x^2} + B\frac{2x}{1 + x^2}$$

Answer: $y = A\frac{1-x^2}{1+x^2} + B\frac{2x}{1+x^2}$