ANSWER on Question #66659 – Math – Differential Equations

QUESTION

True/false. Justify. The normal form of the differential equation

$$y'' - 4xy' + (4x^2 - 1)y = -3e^{x^2} \sin 2x$$

is

$$\left(\frac{d^2v}{dx^2}\right) + v = -3\sin 2x, \quad where \quad v = ye^{-x^2}$$

SOLUTION

$$v = ye^{-x^2} \quad \leftrightarrow \quad y = ve^{x^2}$$

Then

$$y' = \frac{d}{dx}\left(ve^{x^2}\right) = v'e^{x^2} + 2xve^{x^2} = (v' + 2xv)e^{x^2}$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left((v' + 2xv)e^{x^2}\right) = e^{x^2}\frac{d}{dx}(v' + 2xv) + 2x(v' + 2xv)e^{x^2} =$$

$$= (v'' + 2v + 2xv' + 2xv' + 4x^2v)e^{x^2} = (v'' + 2v + 4xv' + 4x^2v)e^{x^2}$$

We substitute the obtained derivatives in the initial equation

$$y'' - 4xy' + (4x^2 - 1)y =$$

$$= (v'' + 2v + 4xv' + 4x^2v)e^{x^2} - 4x(v' + 2xv)e^{x^2} + (4x^2 - 1)ve^{x^2} =$$

$$= (v'' + 2v + 4xv' + 4x^2v - 4x(v' + 2xv) + (4x^2 - 1)v)e^{x^2} =$$

$$= (v'' + 2v + 4xv' + 4x^2v - 4xv' - 8x^2v + 4x^2v - v)e^{x^2} =$$

$$= \left(v'' + \underbrace{2v - v}_{v} + \underbrace{4xv' - 4xv'}_{0} + \underbrace{8x^2v - 8x^2v}_{0}\right)e^{x^2} = (v'' + v)e^{x^2}$$

Hence

$$y'' - 4xy' + (4x^2 - 1)y = -3e^{x^2}\sin 2x \leftrightarrow (v'' + v)e^{x^2} = -3e^{x^2}\sin 2x$$

$$v'' + v = -3\sin 2x \leftrightarrow \frac{d^2v}{dx^2} + v = -3\sin 2x$$

ANSWER: $TRUE$.