Answer on Question 66455 - Math - Differential Equations

The initial value problem

$$\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 0$$

has a unique solution in some interval of the form $-h < x < h$.

Solution The well-known existence and uniqueness theorem states: *if the right-hand $f(x, y)$ of the differential equation $y' = f(x, y)$ is a continuously differentiable function on some domain Ω, then for any initial data $(x_0, y_0) \in \Omega$ we can find a neighbourhood $[x_0 - h, x_0 + h]$ in which there exists an unique solution $y = y(x)$ of the equation such that $y(x_0) = y_0$.*

In our case we have $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$. The function f is continuously differentiable in $\mathbb{R}^2$, since its partial derivatives

$$\frac{\partial f}{\partial x}(x, y) = 2x, \qquad \frac{\partial f}{\partial y}(x, y) = 2y$$

are continuous functions in $\mathbb{R}^2$. Therefore the initial problem

$$\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 0 \tag{1}$$

has a unique solution in an interval $-h < x < h$ with some positive h. In addition, we can find the concrete number h with such property. For instance, we consider the rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \colon |x| \le a, \quad |y| \le b\}$$

and set

$$M = \max_{(x,y)\in R} |f(x, y)|.$$

It is known that then a solution of (1) exists in the interval $-h < x < h$, where

$$h = \min\left\{a, \frac{b}{M}\right\}$$

We have

$$M = \max_{(x,y)\in R} (x^2 + y^2) = a^2 + b^2 \quad \Rightarrow \quad h = \min\left\{a, \frac{b}{a^2 + b^2}\right\}$$

Set $a = 1$ and $b = 1$. Then $h = \min\{1, 0.5\} = 0.5$. Hence, we can state that the solution $y = y(x)$ of (1) exists in the interval $-0.5 < x < 0.5$.

Answer: The statement is true. There exists an unique solution $y = y(x)$ of (1) in the interval $-0.5 < x < 0.5$.