Answer on Question #66446–Math–Algebra

Question

$$\frac{1\cdot 2+\ 2\cdot 3+...+n\left(n+1\right)}{n(n+3)}\geq\frac{\left(n+1\right)}{4},\ \text{ for } n>1. \qquad \textbf{(1)}$$

Solution

First prove that

$$1\cdot 2+\ 2\cdot 3+...+n\left(n+1\right)=\frac{n(n+1)(n+2)}{3}. \qquad \textbf{(2)}$$

Consider the proof of (2) by mathematical induction.

1. Basis of the induction

Show that the statement (2) holds for *n* = 1:

the left-hand side is $1\cdot 2=2$,

the right-hand side is $\frac{1\cdot 2\cdot 3}{3}=2$, hence 2=2 is true.

2. Induction hypothesis

Assume that the statement (2) holds for $n=k$, k>1:

$$1\cdot 2+\ 2\cdot 3+...+k\left(k+1\right)=\frac{k(k+1)(k+2)}{3}. \qquad \textbf{(3)}$$

3. Induction step.

It must then be shown that the statement (2) holds for $n=k+1$, that is,

$$1\cdot 2+\ 2\cdot 3+...+k\left(k+1\right)+(k+1)\left(k+1+1\right)=\frac{(k+1)(k+1+1)(k+1+2)}{3},$$

i.e.

$$1\cdot 2+\ 2\cdot 3+...+k\left(k+1\right)+(k+1)\left(k+2\right)=\frac{(k+1)(k+2)(k+3)}{3}. \qquad \textbf{(4)}$$

Consider the left-hand side of (4):

$1\cdot 2+\ 2\cdot 3+...+k\left(k+1\right)+(k+1)\left(k+2\right)$ = {using the (3) for the first k terms} =

= $\frac{k(k+1)(k+2)}{3}+(k+1)\left(k+2\right)$ = {taking out the common multipliers $(k+1),\left(k+2\right)$} =

= $(k+1)\left(k+2\right)\left(\frac{k}{3}+1\right)=(k+1)\left(k+2\right)\left(\frac{k+3}{3}\right)$ = $\frac{(k+1)\left(k+2\right)(k+3)}{3}$, we deduced to right-hand side of (4).

Thus from the assumption that formula (2) is true for $n=k$, we get that it is also true for $n=k+1$.

4. According to the principle of mathematical induction, formula (2) is proved for all natural numbers.

According to the formula (2), we get that the left-hand side of (1) :

$$\frac{1\cdot 2+\ 2\cdot 3+...+n\left(n+1\right)}{n(n+3)}=\frac{n(n+1)(n+2)}{3n(n+3)}=\frac{(n+1)(n+2)}{3(n+3)},$$

so instead (1) prove the inequality $\frac{(n+1)(n+2)}{3(n+3)}>\frac{(n+1)}{4}$ for $n>1$. **(5)**

Note that

$$\frac{(n+1)(n+2)}{3(n+3)}-\frac{(n+1)}{4}=(n+1)\left(\frac{(n+2)}{3(n+3)}-\frac{1}{4}\right)=(n+1)\left(\frac{4(n+2)-3(n+3)}{12(n+3)}\right)=$$

$$= (n+1)\left(\frac{4n+8-3n-9}{12(n+3)}\right) = (n+1)\left(\frac{n-1}{12(n+3)}\right) > 0$$ for $n > 1$,

so (5) is true, and formula (1) is proved.