Answer on Question #66374 – Math – Linear Algebra

Which of the following sets are convex? Give reason.

Question

(i) $A = \{(x_1, x_2): x_1, x_2 \leq 1;\ x_1, x_2 \geq 0\}$

Solution

We recall the definition of the convex set. A set C is convex if for any $u, v \in C$ the point $\theta u + (1-\theta)v \in C$ for all $\theta \in [0,1]$.

Let $u, v \in A$ and $\theta \in [0,1]$. The vector $u = (u_1, u_2)$ is such that $0 \leq u_1 \leq 1$ and $0 \leq u_2 \leq 1$ and, correspondingly, the vector $v = (v_1, v_2)$ is such that $0 \leq v_1 \leq 1$ and $0 \leq v_2 \leq 1$.

We have to prove that the vector $\theta u + (1-\theta)v = (\theta u_1 + (1-\theta)v_1, \theta u_2 + (1-\theta)v_2) \in A$ for all $\theta \in [0,1]$.

Indeed, for $i = 1$ and $i = 2$: $\theta u_i + (1-\theta)v_i \geq 0$ since $u_i \geq 0, v_i \geq 0, \theta \in [0,1]$ and $\theta u_i + (1-\theta)v_i \leq \theta + (1-\theta)$=1 since $u_i \leq 1, v_i \leq 1, \theta \in [0,1]$. Therefore, $\theta u + (1-\theta)v \in A$ for all $\theta \in [0,1]$ and A is a convex set.

Answer: Convex.

Question

(ii) $B = \{(x_1, x_2):\ x_2 - 3 \geq x_1^2;\ x_1, x_2 \geq 0\}$

Solution

Let us recall that translation preserves convexity of the set. See, for example https://web.stanford.edu/class/ee364a/lectures/sets.pdf

Therefore we translate the set B by vector $(0, -3)$. We come to the set $B' = \{(x_1, x_2):\ x_2 \geq x_1^2;\ x_1, x_2 \geq 0\}$. Let's prove that B' is convex.

Let $u, v \in B'$ and $\theta \in [0,1]$. We have to prove that the vector $\theta u + (1-\theta)v = (\theta u_1 + (1-\theta)v_1, \theta u_2 + (1-\theta)v_2) \in B'$ for all $\theta \in [0,1]$.

Let us consider the inequality:

$$\theta u_1^2 + (1-\theta)v_1^2 \geq (\theta u_1 + (1-\theta)v_1)^2.$$

The following are equivalent:

$$\theta u_1^2 + (1-\theta)v_1^2 \geq (\theta u_1 + (1-\theta)v_1)^2$$

$$\theta u_1^2 + (1-\theta)v_1^2 \geq \theta^2 u_1^2 + 2\theta(1-\theta)u_1 v_1 + (1-\theta)^2 v_1^2$$

$$0 \geq (\theta^2 - \theta)u_1^2 - 2(\theta^2 - \theta)u_1 v_1 + [(1-\theta)^2 - (1-\theta)]v_1^2$$

$$0 \geq (\theta^2 - \theta)u_1^2 - 2(\theta^2 - \theta)u_1 v_1 + (\theta^2 - \theta)v_1^2$$

$$0 \geq (\theta^2 - \theta)(u_1 - v_1)^2$$

The final inequality is true for all θ if $u_1 = v_1$, and if $u_1 \neq v_1$, then the final inequality holds exactly when $\theta \in [0,1]$.

It means that for all $\theta \in [0,1]$ and $u, v \in B'$ we have

$$(\theta u_1 + (1-\theta)v_1)^2 \leq \theta u_1^2 + (1-\theta)v_1^2 \leq \theta u_2 + (1-\theta)v_2.$$

Therefore, $\theta u + (1-\theta)v \in B'$ for all $\theta \in [0,1]$. Then B' and B, correspondingly, are convex.

Answer: Convex.