Answer on Question #66343 – Math – Differential Equations

Question

Show that the wave equation

$$a^2 u_{xx} = u_{tt}$$

can be reduced to the form

$$u_{\xi\eta} = 0$$

by the change of variable

$$\xi = x - at, \ \eta = x + at.$$

Solution

We have the differential equation

$$a^2 u_{xx} = u_{tt}$$

Change the variables x, t by $\xi, \ \eta$:

$$\xi = x - at, \qquad \eta = x + at$$

Note that

$$\xi_x = 1, \qquad \eta_x = 1, \qquad \xi_t = -a, \qquad \eta_t = a$$

Find partial derivatives which are into the equation

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$$
$$u_t = u_\xi \xi_t + u_\eta \eta_t = -a u_\xi + a u_\eta$$
$$u_{xx} = \left(u_\xi + u_\eta\right)_\xi \xi_x + \left(u_\xi + u_\eta\right)_\eta \eta_x = u_{\xi\xi} + u_{\eta\xi} + u_{\xi\eta} + u_{\eta\eta}$$
$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$
$$u_{tt} = \left(-a u_\xi + a u_\eta\right)_\xi \xi_t + \left(-a u_\xi + a u_\eta\right)_\xi \eta_t$$
$$= a^2 u_{\xi\xi} - a^2 u_{\eta\xi} - a^2 u_{\xi\eta} + a^2 u_{\eta\eta} == a^2 u_{\xi\xi} - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta}$$

Substitute the derivatives into the equation

$$a^2 \left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) = a^2 u_{\xi\xi} - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta}$$

Then we get

$$4a^2 u_{\xi\eta} = 0$$

or

$$u_{\xi\eta} = 0$$

Answer: the wave equation $a^2 u_{xx} = u_{tt}$ can be reduced to the form $u_{\xi\eta} = 0$ by the change of variables $\xi = x - at, \ \eta = x + at$.