Answer on Question 66341 - Math - Differential Equations

Find the temperature in a bar of length L with both ends insulated and with initial temperature in the rod being $\sin \frac{\pi x}{L}$.

Solution: We consider the initial boundary value problem for the heat equation

$$u_t = a^2 u_{xx}, \qquad x \in (0, L), \quad t > 0, \tag{1}$$

$$u(0, x) = \sin \frac{\pi x}{L}, \tag{2}$$

$$u_x(t, 0) = u_x(t, L) = 0. \tag{3}$$

We look for a specific type of solution; namely, a product of a function of t only and a function of x only:

$$v(t, x) = T(t)X(x).$$

We substitute this function v into the differential equation (1), and divide by $a^2 v$. This gives

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}.$$

The left-hand side of this equality depends only upon t. The right-hand side is independent of t. It follows that

$$\frac{X''(x)}{X(x)} = -\lambda,$$

where λ is a constant. Then

$$\frac{T'(t)}{a^2 T(t)} = -\lambda.$$

Thus $v(t, x) = T(t)X(x)$ is a solution of the heat equation that satisfies boundary conditions (3) if and only if T and X satisfy the ordinary differential equation

$$T'(t) + \lambda a^2 T(t) = 0, \qquad t \in (0, +\infty) \tag{4}$$

and the boundary value problem

$$X''(x) + \lambda X(x) = 0, \quad x \in (0, L), \tag{5}$$

$$X'(0) = 0, \quad X'(L) = 0. \tag{6}$$

respectively. Since we wish to have $u_x = 0$ for $x = 0$ and $x = L$, we only consider those solutions of the last equation which also satisfy conditions (6).

This homogeneous problem always has the trivial solution $X = 0$, but this is of no use to us. We are interested in cases where this is not the only solution. It is possible for non-negative λ only.

If $\lambda = 0$, then there exists a nonzero solution $X_0 = 1$ of (5), (6). For $\lambda > 0$ we have $X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$. Substituting X into (6) yields

$$
\begin{aligned}
&X'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x;\\
&X'(0) = 0 \quad \Rightarrow \quad C_2 = 0;\\
&X'(L) = 0 \quad \Rightarrow \quad C_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0 \quad \Rightarrow \quad \sin\sqrt{\lambda}L = 0,
\end{aligned}
$$

since both the constants C_1 and C_2 cannot be zero simultaneously. Then X need not be identically zero if and only if $\sin\sqrt{\lambda}L = 0$, that is, if

$$\lambda_n = \frac{\pi^2 n^2}{L^2}, \qquad n = 1, 2, \dots.$$

These values are called the eigenvalues of the problem. The corresponding solutions are

$$X_n(x) = \cos\frac{\pi n x}{L}, \qquad n = 1, 2, \dots.$$

Next, we can solve equation (4) for all eigenvalues:

$$
\begin{aligned}
&T_0' = 0 \quad \Rightarrow \quad T_0(t) = A_0;\\
&T_n' + \frac{a^2\pi^2 n^2}{L^2}T_n = 0 \quad \Rightarrow \quad T_n(t) = A_n e^{-\frac{a^2\pi^2 n^2}{L^2}t}.
\end{aligned}
$$

We attempt to represent the solution u of (1)–(3) as an infinite series

$$u(t,x) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{a^2\pi^2 n^2}{L^2}t}\cos\frac{\pi n x}{L}.$$

We need to determine the coefficients A_n in such a way that initial condition (2) holds. We have

$$u(0,x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{\pi n x}{L} = \sin\frac{\pi x}{L}.$$

The last series is called a Fourier series of function $\sin\frac{\pi x}{L}$. Moreover,

$$A_0 = \frac{1}{L}\int_0^L \sin\frac{\pi x}{L}\,dx, \qquad A_n = \frac{2}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{\pi n x}{L}\,dx.$$

$$A_0 = \int_0^L \sin\frac{\pi x}{L}dx = \frac{L}{\pi}\int_0^L \sin\frac{\pi x}{L} d\left(\frac{\pi x}{L}\right) = -\frac{L}{\pi}\cos\frac{\pi x}{L}\Big|_0^L = \frac{2L}{\pi}.$$

$$
\begin{aligned}
A_n &= \int_0^L \sin\frac{\pi x}{L}\cos\frac{\pi n x}{L}dx = \frac{1}{2}\int_0^L \left(\sin\frac{\pi(n+1)x}{L} + \sin\frac{\pi(n-1)x}{L}\right)dx \\
&= \frac{1}{2}\left(\frac{L}{\pi(n+1)}\int_0^L \sin\frac{\pi(n+1)x}{L} d\left(\frac{\pi(n+1)x}{L}\right)\right. \\
&+ \left.\frac{L}{\pi(n-1)}\int_0^L \sin\frac{\pi(n-1)x}{L} d\left(\frac{\pi(n-1)x}{L}\right)\right) \\
&= \frac{1}{2}\left(-\frac{L}{\pi(n+1)}\cos\frac{\pi(n+1)x}{L}\Big|_0^L - \frac{L}{\pi(n-1)}\cos\frac{\pi(n-1)x}{L}\Big|_0^L\right) \\
&= (1-(-1)^{n+1})\left(\frac{L}{\pi(n+1)} + \frac{L}{\pi(n-1)}\right) = \frac{2(1-(-1)^{n+1})nL}{\pi(n^2-1)} \\
&= \begin{cases} \frac{8kL}{\pi(4k^2-1)} & \text{if} \quad n = 2k \\ 0 \quad \text{if} \quad n = 2k-1 \end{cases} .
\end{aligned}
$$

Answer:

$$
u(t,x) = \frac{2L}{\pi} + \sum_{k=1}^{\infty} \frac{8kL}{\pi(4k^2-1)} e^{-\frac{4a^2\pi^2k^2}{L^2}t} \cos\frac{2\pi k x}{L}.
$$