Answer on Question #66340 – Math – Differential Equations

Question

Using the method of separation of variables, solve

$$u_{xt} = e^{-t}\cos x \quad \text{(i)}$$

when

$$u(x,0) = 0,$$

$$\frac{\partial u}{\partial t}(0,t) = 0.$$

Solution

We shall search for the solution of (i) in the following form:

$$u(x,t) = X(x)T(t).$$

Substituting it in the equation (i) and the initial conditions we obtain

$$X'(x)T'(t) = e^{-t}\cos x$$

$$X(x)T(0) = 0, \qquad X(0)T'(t) = 0.$$

We remark that $X(x) \neq 0$, because $u(x,t)$ with $X = 0$ is not a solution of the equation. Therefore, $T(0) = 0$. Further, $T'(t)$ cannot be equal to zero, since it contradicts the equation

$$X'(x)T'(t) = e^{-t}\cos x.$$

Therefore,

$$X(0) = 0.$$

So we obtain two initial value problems:

$$X'(x) = \cos x, \;\; X(0) = 0 \;\; (1)$$

and

$$T'(t) = e^{-t}, \; T(0) = 0. \quad (2)$$

Solving the problem (1) we obtain

$$X'(x) = \cos x,$$

$$X(x) = \sin x + C_1.$$

$$X(0) = C_1 = 0.$$

So, $X(x) = \sin x$ is a solution of the problem (1).

Solving the problem (2) we obtain

$$T'(t) = e^{-t},$$

$$T(t) = -e^{-t} + C_2.$$

$$T(0) = -1 + C_2 = 0 \rightarrow \ C_2 = 1.$$

Then

$$T(t) = -e^{-t} + 1$$

is a solution of the problem (2).

Therefore, the solution of the original problem (i) satisfying the initial conditions is

$$u(x,t) = X(x)T(t) = (1 - e^{-t}) \sin x = -e^{-t} \sin x + \sin x\,.$$

Answer: $u(x,t) = -e^{-t} \sin x + \sin x\,.$