Answer on Question #66335 – Math – Differential Equations

Question

Solve the following equation by Jacobi's method

$$x^2\frac{\partial u}{\partial x}-\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2=0$$

Solution

$x^2\frac{\partial u}{\partial x}-\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2=0$ (1)

$f(x,z,u_x,u_z)=x^2u_x-u_x^2-u_z^2$

$f_{u_x}=x^2-2u_x,$

$f_{u_z}=-2u_z,$

$f_x=2xu_x,$

$f_z=0.$

$\frac{dx}{f_{u_x}}=\frac{dz}{f_{u_z}}=\frac{du_x}{-f_x}=\frac{du_z}{-f_z}$

$\frac{dx}{x^2-2u_x}=\frac{dz}{-2u_z}=\frac{du_x}{-2xu_x}=\frac{du_z}{0}.$

Now

$\frac{dz}{-2u_z}=\frac{du_z}{0},$

$-2u_z du_z=0,$

$du_z^2=0,$

$u_z^2=C_1,$

$u_z=C,$

hence

$$u=Cz+\varphi(x)\ (2)$$

$u_x=\varphi'(x),$

$u_z = C.$

Now

$\frac{dx}{x^2-2u_x} = \frac{du_x}{-2xu_x},$

$\frac{2xdx}{x^2-2u_x} = \frac{du_x}{-u_x},$

$\frac{dx^2}{x^2-2u_x} = -\frac{du_x}{u_x},$

$x^2 = s$

$$\frac{ds}{s-2u_x} = -\frac{du_x}{u_x}$$

$$\frac{u_x}{2u_x - s} = \frac{du_x}{ds}$$

$u_x = s \cdot t(s),$

$\frac{du_x}{ds} = t(s) + st'(s),$

$\frac{st(s)}{2st(s)-s} = t(s) + st'(s),$

$\frac{t(s)}{2t(s)-1} = t(s) + st'(s),$

$\frac{t(s)}{2t(s)-1} - t(s) = st'(s),$

$\frac{t(s)-2t^2(s)+t(s)}{2t(s)-1} = st'(s),$

$\frac{2t-1}{2t-2t^2} dt = \frac{ds}{s},$

$\frac{2t-1}{2t(1-t)} dt = \frac{ds}{s},$

$\frac{1-2t}{2t(t-1)} dt = \frac{ds}{s},$

$$\int \frac{1-2t}{2t(t-1)} dt = \int \frac{ds}{s}$$

$$\int \left(-\frac{1}{2t} - \frac{1}{2(t-1)}\right) dt = \int \frac{ds}{s}$$

$$-\frac{1}{2}ln|t|-\frac{1}{2}ln\,|t-1|=ln|s|-lnC_2$$

$$s=\frac{C_2}{\sqrt{t(t-1)}},$$

$$t^2-t+\frac{1}{4}=\frac{C_2^2}{s^2}+\frac{1}{4},$$

$$t-\frac{1}{2}=\sqrt{\frac{C_2^2}{s^2}+\frac{1}{4}}\ \text{ or }\ t-\frac{1}{2}=-\sqrt{\frac{C_2^2}{s^2}+\frac{1}{4}},$$

$$t=\sqrt{\frac{C_2^2}{s^2}+\frac{1}{4}}+\frac{1}{2}\ \text{ or }\ t=-\sqrt{\frac{C_2^2}{s^2}+\frac{1}{4}}+\frac{1}{2},$$

$$u_x=ts=\sqrt{C_2^2+\frac{s^2}{4}}+\frac{s}{2}=\sqrt{C_2^2+\frac{x^4}{4}}+\frac{x^2}{2}\text{ or }u_x=ts=-\sqrt{C_2^2+\frac{x^4}{4}}+\frac{x^2}{2}$$

$$u=\int\sqrt{C_2^2+\frac{x^4}{4}}dx+\frac{x^3}{6}+\mu(z)\text{ or }u=-\int\sqrt{C_2^2+\frac{x^4}{4}}dx+\frac{x^3}{6}+\mu(z)\ (3)$$

Comparing (2) and (3) one gets

$$u=\int\sqrt{C_2^2+\frac{x^4}{4}}dx+\frac{x^3}{6}+Cz\ \text{ or }\ u=-\int\sqrt{C_2^2+\frac{x^4}{4}}dx+\frac{x^3}{6}+Cz$$

Check the case of $\frac{\partial u}{\partial x}=u_x=0$. It follows from equation (1) that $\left(\frac{\partial u}{\partial z}\right)^2=0$, hence $\frac{\partial u}{\partial z}=0$. One got $\frac{\partial u}{\partial x}=0$ and $\frac{\partial u}{\partial z}=0$ simultaneously, hence $u=C_1$.

Answer:

$$u=\int\sqrt{C_2^2+\frac{x^4}{4}}dx+\frac{x^3}{6}+Cz;\ \ u=-\int\sqrt{C_2^2+\frac{x^4}{4}}dx+\frac{x^3}{6}+Cz;\ u=C_1.$$